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The Generalized Wronskian Solution to a Negative KdV-mKdV Equation

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A negative KdV-mKdV hierarchy is presented through the KdV-mKdV operator. The generalized Wronskian solution to the negative KdV-mKdV equation is obtained. Some soliton-like solutions and a complexiton solution are presented explicitly as examples.

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Recently, the negative integrable equations have attracted great interest. For example, in Ref. [1] Damianou discussed the negative Toda hierarchy and its master symmetries and rational Poisson brackets. In Refs. [2,3], Ji *et al.* considered the negative AKNS hierarchy and obtained its soliton solutions. In Refs. [4,5] the authors investigated the negative KdV equation and presented its soliton solutions and rational solutions. Many classical methods such as the Hirota method, the Wronskian and generalized Wronskian technique and the Darboux transformation are applied to find the exact solutions to the negative equations. Among these methods, the Wronskian and generalized Wronskian technique is widely used because the negative equations can usually be bilinearlized through the same dependent variable transformation as that of positive equations. In general, the Wronski determinant is constructed by the nonlinear integrable equation's spectral functions whose derivative with time can be written in the form including no spectral parameter explicitly. However, for some negative equations, their spectral function's derivative does not have this feature and this causes trouble for the use of the Wronskian technique. Thus the first task in dealing with these negative equations is to seek a suitable transformation of the spectral functions to make the derivative with time have a suitable form.

It is well known that when $u = v_x$ we can derive the sine-Gordon equation from the inverse mKdV operator,

$$F = \partial^2 + 4u^2 + 4u_x\partial^{-1}.$$

From the inverse KdV operator

$$T = \partial^2 + 4u + 2u_x\partial^{-1},$$

one can derive the associated Camassa–Holm equation.[4] The generalized operator

$$S = \partial^2 + 4u(\alpha + \beta u) + 2u_x\partial^{-1}(\alpha + 2\beta u), \qquad (1)$$

gives the mixed KdV-mKdV equation and this hierarchy is known as the reduction of AKNS hierarchy by taking $q = \alpha + \beta u,\ r = -u$.[6–10] Now we use the inverse operator to produce a new hierarchy as follows:

$$S^n u_{t_n} = u_x. \qquad (2)$$

The first equation in hierarchy is the negative mixed KdV-mKdV equation

$$u_{txx} + 4u(\alpha+\beta u)u_t + 2u_x\partial^{-1}(\alpha u_t + 2\beta u u_t) = u_x. \qquad (3)$$

Under the condition $\beta \neq 0$ we use an independent variable transformation[6]

$$u = \frac{1}{2}\alpha\beta^{-1} + i\beta^{-1/2}\Big(\ln\frac{f^*}{f}\Big)_x, \qquad (4)$$

$(i = \sqrt{-1})$ to bilinearize it to

$$D_x^2 f^* \cdot f = 0, \qquad (5)$$

$$(D_x^2 D_t - bD_t - D_x)f^* \cdot f = 0, \quad b = \alpha^2\beta^{-1}, \qquad (6)$$

where the superscript * means the complex conjugate, and D is the bilinear operator defined as

$$D_x^m D_t^n g(x,t)\cdot f(x,t) = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n g(x,t)f(x',t')|_{x'=x,t'=t}.$$

In order to construct the Wronski determinant we first list several Lemmas.

Lemma 1.[2,3,14] Assuming that H is an $n\times(n-2)$ matrix and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are n-dimensional vectors, then we have the following determinantal identity

$$|Hab||Hcd| - |Hac||Hbd| + |Had||Hbc| = 0.$$

Lemma 2. Assuming that H is an $n\times(n-3)$ matrix and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}$ are n-dimensional vectors, then we have the following determinantal identity

$$|Habc||Hefd| - |Habd||Hefc| + |Hacd||Hefb| - |Hbcd||Hefa| = 0.$$

Proof. Consider the $2n\times 2n$ determinant

$$\begin{vmatrix} H & O & 0 & 0 & a & b & c & d \\ O & H & e & f & a & b & c & d \end{vmatrix}.$$

Expanding it in the first n rows we can obtain the result immediately.

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Lemma 3.[5] Assuming that H is an $n\times n$ matrix, $\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\ldots,\boldsymbol{\beta}_n$ are the columns of another $n\times n$ matrix, then we have the following formula

$$(\mathrm{tr}H)|\beta_1,\beta_2,\ldots,\beta_n|=\sum_{j=1}^{n}|\beta_1,\beta_2,\ldots,H\beta_j,\ldots,\beta_n|.$$

Take the Wronski determinant

$$f=|\varphi,\varphi',\ldots,\varphi^{(N-1)}|, \tag{7}$$

where $\boldsymbol{\varphi}=(\varphi_1,\varphi_2,\ldots,\varphi_N)^T$ and superscript (j) means the jth derivative with respect to x. The spectral function vector $\boldsymbol{\varphi}$ satisfies

$$\boldsymbol{\varphi}_{xx}=\frac{A^2}{4}\boldsymbol{\varphi},\ \boldsymbol{\varphi}_t=(A^2-bI)^{-1}\boldsymbol{\varphi}_x,\ \boldsymbol{\varphi}^*=-2iA^{-1}\boldsymbol{\varphi}_x, \tag{8}$$

where A is an $N\times N$ constant matrix, I is the $N\times N$ unit matrix and the subscript means the partial derivative. Equation (8) comes from the Hirota method, which is omitted here.

We put up a transformation of the spectral function of the negative KdV-mKdV equation:

$$\boldsymbol{\varphi}=e^{\sqrt{b}x/2}\boldsymbol{\phi}.$$

It is easy to know that Eq. (8) can be changed into a new form:

$$\boldsymbol{\phi}_{xx}=\frac{A^2-bI}{4}\boldsymbol{\phi}-\sqrt{b}\boldsymbol{\phi}_x, \tag{9}$$

$$\boldsymbol{\phi}_t=(A^2-bI)^{-1}(\frac{\sqrt{b}}{2}\boldsymbol{\phi}+\boldsymbol{\phi}_x), \tag{10}$$

$$\boldsymbol{\phi}^*=-2iA^{-1}(\frac{\sqrt{b}}{2}\boldsymbol{\phi}+\boldsymbol{\phi}_x). \tag{11}$$

From these equations, we immediately obtain

$$\partial_t\boldsymbol{\phi}^{(j)}=B\boldsymbol{\phi}^{(j)}+\frac{1}{4}\boldsymbol{\phi}^{(j-1)},\ B=-\frac{\sqrt{b}}{2}(A^2-bI)^{-1}.$$

This relation ensures that we can take the derivative with time of the determinants constructed by $\boldsymbol{\phi}$.

In the following, we use the Nimmo and Freeman's[11,12] abbreviated notation $|\widehat{N-1}|$ to denote the Wronskian:

$$|\widehat{N-1}|=|\phi,\phi',\ldots,\phi^{(N-1)}|.$$

Further, we introduce more notations to denote the determinants whose first column is not the zero-order derivative of ϕ and the determinants constructed by both vector functions $\boldsymbol{\varphi}$ and $\boldsymbol{\phi}$. For instance,

$$|-1,\widehat{N-2}|=|\phi^{(-1)},\phi,\phi',\ldots,\phi^{(N-2)}|,$$
$$|\widetilde{N}|=|\phi',\phi'',\ldots,\phi^{(N)}|,$$
$$|\overline{-1},\widehat{N-2}|=|\varphi^{(-1)},\phi,\phi',\ldots,\phi^{(N-2)}|,$$

where negative superscript means the integral with respect to variable x. Let

$$F=|\widehat{N-1}|,\ G=|\overline{-1},\widehat{N-2}|.$$

Obviously we have

$$f=e^{N\sqrt{b}x/2}F,\ f^*=Ke^{(N-1)\sqrt{b}x/2}G,\ K=\frac{(-i)^N}{2^N}|A|.$$

Then the bilinear equations (5) and (6) can be transformed into

$$(\frac{b}{4}-\sqrt{b}D_x+D_x^2)G\cdot F=0, \tag{12}$$

$$(\frac{\sqrt{b}}{2}-D_x-\frac{3}{4}bD_t-\sqrt{b}D_xD_t+D_x^2D_t)G\cdot F=0. \tag{13}$$

For convenience we use Δ_1 and Δ_2 to represent the left hand side of these two equations respectively.

By using lemma 3, we simplify Δ_2 to

$$\begin{aligned}\Delta_2=&\frac{3\sqrt{b}}{8}GF+\frac{3b}{16}(G\widetilde{F}-\widetilde{G}F)\\&+\frac{1}{2}D_xG\cdot F-\frac{1}{2}\widetilde{G_x}F_x+\frac{1}{2}G_x\widetilde{F_x}+\frac{\sqrt{b}}{4}(\widetilde{G}F_x\\&-G\widetilde{F_x}+G_x\widetilde{F}-\widetilde{G_x}F)-\frac{1}{2\sqrt{b}}D_x^2G\cdot F\\&+\frac{1}{4}(\widetilde{G_{xx}}F-G_{xx}\widetilde{F}+\widetilde{G}F_{xx}-G\widetilde{F_{xx}})\\&+e^{\sqrt{b}x/2}[-\frac{3\sqrt{b}}{8}\widehat{G}F-\frac{1}{2}\widehat{G_x}F-\frac{1}{\sqrt{b}}\widehat{G_x}F_x\\&+\frac{1}{2}\widehat{G}F_x+\frac{1}{2\sqrt{b}}\widehat{G}F_{xx}+\frac{1}{2\sqrt{b}}\widehat{G_{xx}}F],\end{aligned} \tag{14}$$

where

$$\widetilde{F}=|-1,\widetilde{N-1}|,\ \widetilde{F_x}=|-1,\widetilde{N-2},N|,$$
$$\widetilde{F_{xx}}=|-1,\widetilde{N-3},N-1,N|+|-1,\widetilde{N-2},N+1|,$$
$$\widehat{G}=|-1,\widehat{N-2}|,\ \widehat{G_x}=|-1,\widehat{N-3},N-1|,$$
$$\widehat{G_{xx}}=|-1,\widehat{N-4},N-2,N-1|+|-1,\widehat{N-3},N|,$$
$$\widetilde{G}=|\overline{-1},-1,\widetilde{N-2}|,\widetilde{G_x}=|\overline{-1},-1,\widetilde{N-3},N-1|,$$
$$\widetilde{G_{xx}}=|\overline{-1},-1,\widetilde{N-4},N-2,N-1|$$
$$+|\overline{-1},-1,\widetilde{N-3},N|.$$

Further, we set

$$S_1=|\widehat{N-3},N-1,N|,\quad S_2=|\widehat{N-2},N+1|,$$
$$T_1=|\overline{-1},\widehat{N-4},N-2,N-1|,$$
$$T_2=|\overline{-1},\widehat{N-3},N|,$$
$$\widetilde{S_1}=|-1,\widetilde{N-3},N-1,N|,$$
$$\widetilde{S_2}=|-1,\widetilde{N-2},N+1|,$$
$$\widetilde{T_1}=|\overline{-1},-1,\widetilde{N-4},N-2,N-1|,$$
$$\widetilde{T_2}=|\overline{-1},-1,\widetilde{N-3},N|,$$
$$\widehat{T_1}=|-1,\widehat{N-4},N-2,N-1|,$$
$$\widehat{T_2}=|-1,\widehat{N-3},N|.$$

Then from a series of identities derived from lemma 3:

$$(\mathrm{tr}\frac{A^2-bI}{4})F = -S_1 + S_2 + \sqrt{b}F_x, \tag{15}$$

$$(\mathrm{tr}\frac{A^2-bI}{4})\widetilde{F} = -\widetilde{S_1} + \widetilde{S_2} + \sqrt{b}(F + \widetilde{F_x}), \tag{16}$$

$$(\mathrm{tr}\frac{A^2-bI}{4})G = -T_1 + T_2 + \sqrt{b}G_x - \frac{b}{4}G, \tag{17}$$

$$(\mathrm{tr}\frac{A^2-bI}{4})\widetilde{G} = -\frac{b}{4}\widetilde{G} - \widetilde{T_1} + \widetilde{T_2} + \sqrt{b}(G + \widetilde{G_x}) - \frac{\sqrt{b}}{2}e^{\sqrt{b}x/2}\widehat{G}, \tag{18}$$

$$(\mathrm{tr}\frac{A^2-bI}{4})\widehat{G} = -\widehat{T_1} + \widehat{T_2} + \sqrt{b}\widehat{G_x}, \tag{19}$$

$$\begin{aligned}(\mathrm{tr}\frac{A^2-bI}{4})|\overline{-1}, \widetilde{N-1}| &= -\frac{b}{4}|\overline{-1}, \widetilde{N-1}| \\ &- |\overline{-1}, \widetilde{N-3}, N-1, N| + |\overline{-1}, \widetilde{N-2}, N+1| \\ &+ \sqrt{b}|\overline{-1}, \widetilde{N-2}, N| + \frac{\sqrt{b}}{2}e^{\sqrt{b}x/2}F,\end{aligned} \tag{20}$$

we deduce five relations

$$\begin{aligned}&\frac{b}{4}GF - \sqrt{b}D_xG\cdot F + (T_1 - T_2)F \\ &- (S_1 - S_2)G = 0,\end{aligned} \tag{21}$$

$$\begin{aligned}&-\sqrt{b}GF + \sqrt{b}(G_x\widetilde{F} - G\widetilde{F_x}) - \frac{b}{4}G\widetilde{F} \\ &+ G(\widetilde{S_1} - \widetilde{S_2}) - (T_1 - T_2)\widetilde{F} = 0,\end{aligned} \tag{22}$$

$$\begin{aligned}&\sqrt{b}GF + \sqrt{b}(\widetilde{G_x}F - \widetilde{G}F_x) - \frac{b}{4}\widetilde{G}F + \widetilde{G}(S_1 - S_2) \\ &- (\widetilde{T_1} - \widetilde{T_2})F - \frac{\sqrt{b}}{2}e^{\sqrt{b}x/2}\widehat{G}F = 0,\end{aligned} \tag{23}$$

$$\widehat{G}(S_1 - S_2 - \sqrt{b}F_x) - (\widehat{T_1} - \widehat{T_2} - \sqrt{b}\widehat{G_x})F = 0, \tag{24}$$

$$\begin{aligned}&\widehat{G}(|\overline{-1}, \widetilde{N-3}, N-1, N| - |\overline{-1}, \widetilde{N-2}, N+1| \\ &- \sqrt{b}|\overline{-1}, \widetilde{N-2}, N| + \frac{b}{4}|\overline{-1}, \widetilde{N-1}|) \\ &- |\overline{-1}, \widetilde{N-1}| \\ &(|-1, \widehat{N-4}, N-2, N-1| - |-1, \widehat{N-3}, N| \\ &- \sqrt{b}|-1, \widehat{N-3}, N-1|) - \frac{\sqrt{b}}{2}e^{\sqrt{b}x/2}\widehat{G}F = 0.\end{aligned} \tag{25}$$

Substituting Eq. (21) to Eq. (12) yields

$$\Delta_1 = 2(T_2F - G_xF_x + S_1G), \tag{26}$$

which is zero according to lemma 1. Inserting Eqs. (12), (22) and (23) into Eq. (14) yields

$$\begin{aligned}\Delta_2 = &\frac{b}{8}(G\widetilde{F} - \widetilde{G}F) - \frac{1}{2}(\widetilde{G_x}F_x - G_x\widetilde{F_x}) + \frac{\sqrt{b}}{2}(\widetilde{G}F_x \\ &- G\widetilde{F_x} + G_x\widetilde{F} - \widetilde{G_x}F) + \frac{1}{2}(S_2\widetilde{G} - \widetilde{S_2}G) - \frac{1}{2}(T_1\widetilde{F} \\ &- \widetilde{T_1}F) + e^{\sqrt{b}x/2}[-\frac{\sqrt{b}}{4}\widehat{G}F + \frac{1}{2\sqrt{b}}\widehat{G_{xx}}F - \frac{1}{2}\widehat{G_x}F \\ &+ \frac{1}{2}\widehat{G}F_x - \frac{1}{\sqrt{b}}\widehat{G_x}F_x + \frac{1}{2\sqrt{b}}\widehat{G}F_{xx}].\end{aligned} \tag{27}$$

Utilizing lemma 1 to transform the first, third, fourth and fifth terms of Δ_2 we obtain

$$\begin{aligned}\Delta_2 = &-\frac{1}{2}(\widetilde{G_x}F_x - G_x\widetilde{F_x}) + \frac{b}{8}\widehat{G}|\overline{-1}, \widetilde{N-1}| \\ &+ \frac{1}{2}|\overline{-1}, \widetilde{N-1}|(-|-1, \widehat{N-4}, N-2, N-1| \\ &+ \sqrt{b}|-1, \widehat{N-3}, N-1|) + \frac{1}{2}|-1, \widehat{N-2}| \\ &(|\overline{-1}, \widetilde{N-2}, N+1| - \sqrt{b}|\overline{-1}, \widetilde{N-2}, N|) \\ &+ e^{\sqrt{b}x/2}[\frac{1}{2\sqrt{b}}\widehat{G}(F_{xx} - \frac{b}{2}F + \sqrt{b}F_x) \\ &- \frac{1}{2\sqrt{b}}\widehat{G_x}(2F_x + \sqrt{b}F) + \frac{1}{2\sqrt{b}}\widehat{G_{xx}}].\end{aligned}$$

Finally, inserting Eqs. (24) and (25) into it and applying lemma 2 we find that Δ_2 does equal zero. Thus we complete the proof.

Now we present some examples of exact solutions of the negative KdV-mKdV equation. Since Ma[13] proposed the concept of the complexiton, the generalized Wronskian technique has found wider use. In 2008, Chen *et al.*[14] used the matrix technique to deal with the generalized Wronskian, which makes the procedure simple and clear. In the following we use this technique to construct concrete exact solutions to the negative KdV-mKdV equation. We first set

$$Q = Q(x,t) = \frac{A}{2}x + \frac{A(A^2-bI)^{-1}}{2}t.$$

Then we have the general solution of Eq. (8) as follows:

$$\boldsymbol{\varphi} = e^{i\frac{\pi}{4}}e^{Q}\boldsymbol{C} + e^{-i\frac{\pi}{4}}e^{-Q}\boldsymbol{D}, \tag{28}$$

where $\boldsymbol{C}$ and $\boldsymbol{D}$ are constant vectors.

If the matrix A adopts the diagonal form

$$A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N),$$

Eq. (28) gives rise to a soliton-like solution to the negative KdV-mKdV equation. In fact, taking

$$\boldsymbol{C} = (1, 1, \ldots, 1)^T,\ \boldsymbol{D} = (1, -1, \ldots, (-1)^{N+1})^T,$$

we can compute the determinant f easily and determine the solutions according to the formula (4). In the case $N = 1$, the single-soliton-like solution is

$$u = -\frac{\alpha\beta^{-1}}{2} + \frac{\lambda_1\beta^{-1/2}}{\cosh\xi_1}, \tag{29}$$

where $\xi_1 = \lambda_1 x + \lambda_1(\lambda_1^2 - b)^{-1}t$. In the case $N = 2$, we obtain a two-soliton-like solution as follows:

$$u = -\frac{\alpha\beta^{-1}}{2} - \beta^{-1/2}(\lambda_1 - \lambda_2)^2 \cdot \frac{(\lambda_2 - \lambda_1)\chi_1 + (\lambda_2 + \lambda_1)\chi_2}{(\lambda_1 + \lambda_2)^2\varpi_1 + (\lambda_1 - \lambda_2)^2\varpi_2}, \quad (30)$$

where

$$\xi_j = \lambda_j x + \lambda_j(\lambda_j^2 - b)^{-1}t, \quad j = 1, 2$$
$$\chi_1 = \sinh\frac{\xi_1 - \xi_2}{2}\sinh\frac{\xi_1 + \xi_2}{2}, \ \varpi_1 = \cosh^2\frac{\xi_1 - \xi_2}{2},$$
$$\chi_2 = \cosh\frac{\xi_1 - \xi_2}{2}\cosh\frac{\xi_1 + \xi_2}{2}, \varpi_2 = \sinh^2\frac{\xi_1 + \xi_2}{2}.$$

If matrix A is

$$A = \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix} = \mu I + \nu H,$$

where

$$H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

I means the second-order unit matrix. We can deduce that

$$Q = \Xi I + \Upsilon H,$$

where

$$\Xi = \frac{\mu}{2}x + \frac{\mu - k\nu}{2(\mu^2 - \nu^2 - b)(1 + k^2)}t,$$
$$\Upsilon = \frac{\nu}{2}x + \frac{k\mu + \nu}{2(\mu^2 - \nu^2 - b)(1 + k^2)}t,$$
$$k = \frac{2\mu\nu}{b + \nu^2 - \mu^2}.$$

The calculation gives a complexiton solution to the negative KdV-mKdV equation as follows:

$$u = -\frac{\alpha\beta^{-1}}{2} - 2\beta^{-1/2}\mu\nu \cdot \frac{\nu\cos(2\Upsilon)\cosh(2\Xi) - \mu\sin(2\Upsilon)\sinh(2\Xi)}{[\mu\sin(2\Upsilon)]^2 + [\nu\cosh(2\Xi)]^2}.$$

In summary, we have discussed a negative KdV-mKdV equation and constructed its generalized Wronskian solution. To take the derivative with time of Wronski determinants we introduce a transformation of spectral function. This transformation makes $\partial_t\phi^{(j)}$ still have the form of $\phi^{(j)}$ and then the derivative of determinants can be obtained via lemma 3. We also exhibit some examples of concrete exact solutions. The technique used to treat the parameter in the derivative with time can be applied into other negative integrable systems.

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