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Wavelet Phase Synchronization of Fractional-Order Chaotic Systems *

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Nowadays, fractional-order systems are attracting more and more attention. There are several ways available for analyzing fractional-order systems, among which wavelet transform is an efficient method for analyzing system dynamics in both time and frequency domains. We investigate the wavelet phase synchronization employing wavelet transform to explore the phase synchronization behaviors of fractional-order chaotic oscillators. We analyze in detail the synchronization behaviors with changes to the coupling strength, the central frequency Ω_0, and the time scale of the wavelet.

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The chaotic dynamics of fractional order systems began to attract more and more attention recently. Fractional calculus has a history of more than 300 years,[1] the applications of which to physics and engineering are just a recent focus of interest. Many systems are known to display fractional-order dynamics, such as viscoelastic systems,[2] dielectric polarization,[3] electrode-electrolyte polarization,[4] and electromagnetic waves.[5] Many investigations were devoted to dynamics[6−15] of fractional-order systems. Specifically, in Ref. [6] chaos and hyperchaos in the fractional order Rössler equations were studied, in which we showed that chaos can exist in the fractional order Rössler equation with order as low as 2.4, and hyperchaos can exist in the fractional order Rössler hyperchaos equation with order as low as 3.8.

In the study of chaotic systems, synchronization is a focus of interest.[16−18] Besides the identical (complete) synchronization of chaotic systems, some other types of synchronization also have very interesting cooperative behaviors of chaotic systems, including phase synchronization,[19,20] lag synchronization,[21] projective synchronization,[22] and wavelet phase synchronization/time-scale synchronization of chaotic oscillators.[23−26]

In Ref. [27] we have studied the synchronization of fractional order chaotic systems, and since then, the synchronization of fractional order chaotic systems has begun to attract the attention of some researchers, see e.g. Ref. [28]. However, in the literature, the authors are mostly concerned with the identical synchronization of fractional order chaotic systems. In Ref. [29] we studied the phase and lag synchronization of coupled fractional order chaotic oscillators. In Ref. [30] we investigated the projective synchronization of fractional order chaotic systems. However, to our knowledge, wavelet phase synchronization in fractional order chaotic systems has not been discussed yet. In this Letter, we address it in detail.

In Refs. [23-26] the researchers proposed to use a wavelet approach to investigate the synchronization of chaotic systems. It was approved that synchronization can be detected by introducing a complex wavelet transform, as demonstrated in Ref. [23], $W(a,t) = |W(a,t)|\exp(i\varphi(a,t))$, where $\varphi(a,t)$ is the phase defined. The corresponding condition for the synchronization between chaotic systems with phases $\varphi_1(a,t)$ and $\varphi_2(a,t)$ on a time scale a is $|\varphi_1(a,t) - \varphi_2(a,t)| <$ const. In the present study, this wavelet approach is applied to the coupled fractional-order chaotic systems. As in many studies of phase synchronization, the chaotic system is selected as the Rössler oscillator.

First, let us build the system. Our system consists of two coupled fractional-order Rössler oscillators. There are several definitions of fractional derivatives in the literature.[1] The best known may be the Riemann–Liouville definition, which is given by

$$\frac{d_\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \quad (1)$$

where Γ is the gamma function and $n-1 \le \alpha < n$. Upon considering all the initial conditions to be zero, the Laplace transform of the Riemann–Liouville fractional derivative is $L\{\frac{d^\alpha f(t)}{dt^\alpha}\} = s^\alpha L\{f(t)\}$. Thus, the fractional integral operator of order α can be represented by the transfer function $F(s) = 1/s^\alpha$ in the frequency domain. The standard definition of fractional differintegral does not allow direct implementation of the fractional operators in time-domain simulations. An efficient method to circumvent this problem is to approximate fractional operators by using stan-

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dard integer order operators. In the following simulations, we use the approximation method proposed in Ref. [31], which was also adopted in many studies. In Table 1 of Ref. [7], the authors presented the approximations for $1/s^q$ with $q = 0.1 - 0.9$ in steps 0.1 with errors of approximately 2 dB. We will mainly use these approximations in the following simulations. In our model, the conventional derivative is replaced by a fractional derivative as follows:

$$\begin{aligned}
\frac{d^\alpha x_{1,2}}{dt^a} &= -\omega_{1,2} y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}),\\
\frac{d^\alpha y_{1,2}}{dt^a} &= \omega_{1,2} x_{1,2} + b y_{1,2},\\
\frac{d^\alpha z_{1,2}}{dt^a} &= 0.2 + z_{1,2}(x_{1,2} - 10),
\end{aligned} \tag{2}$$

where α is the fractional-order, b is a constant, and $\omega_{1,2}$ are system parameters for the oscillators 1 and 2, which control the frequency. The parameter ε is the coupling strength between the two oscillators. From Ref. [6], we set $\alpha = 0.9$, $b = 0.35$, and $\omega_{1,2} \approx 1$, then the chaotic behavior is found. The chaotic attractor is shown in Fig. 1.

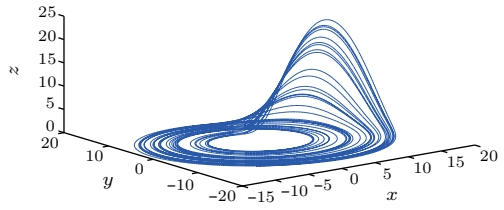


Fig. 1. Chaotic attractor of the fractional-order Rössler oscillator.

Next, let us consider to use the continuous Morlet-wavelet to obtain the wavelet transform of chaotic time series,[23,24] so that we can analyze the wavelet phase synchronization of this coupled system.

$$W(a, t_0) = \int_{-\infty}^{+\infty} x(t)\psi^*_{a,t_0}(t)dt, \tag{3}$$

where $\psi_{a,t_0}(t)$ is the wavelet function related to the mother-wavelet $\psi_0(t)$ as $\psi_{a,t_0}(t) = \frac{1}{\sqrt{s}}\psi_0(\frac{t-t_0}{s})$. The time scale a corresponds to the width of the wavelet function $\psi_{a,t_0}(t)$, and t_0 is the time shift of the wavelet. The Morlet-wavelet[32]

$$\psi_0(\eta) = \frac{1}{\sqrt[4]{\pi}} \exp(j\Omega_0\eta) \exp(\frac{-\eta^2}{2}) \tag{4}$$

is regarded as a mother-wavelet function. The wavelet surface function is

$$W(a, t) = |W(a, t)| \exp(i\varphi(a, t)). \tag{5}$$

The phase $\varphi(a, t) = \arg W(a, t)$.

We define the mean frequency difference as $\Delta\Omega = \langle\dot{\phi}_1(a, t) - \dot{\phi}_2(a, t)\rangle$,[19] where $\langle\cdot\rangle$ denotes the average over time. When $\Delta\Omega \approx 0$, the phase difference does not grow with time, i.e. $|\varphi_1(a, t) - \varphi_2(a, t)| < \text{const}$, thus the phase synchronization condition is satisfied.

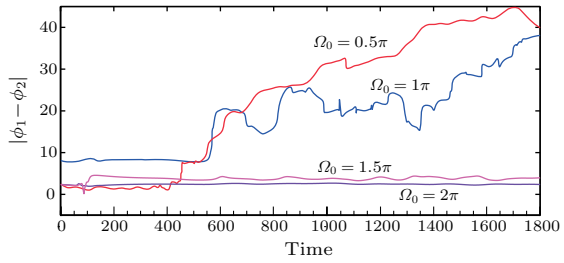


Fig. 2. Phase differences calculated for different central-frequency values.

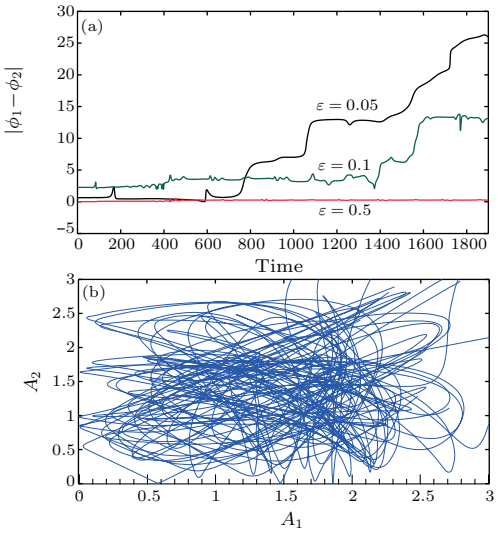


Fig. 3. (a) Phase differences of the coupled fractional-order Rössler equations at time scale $a = 5$. (b) The correlation between the amplitude A_1, A_2. Here $a = 5$, $\varepsilon = 0.5$.

In our simulation, the time series used for wavelet transform are generated by the two coupled Rössler equations. In Ref. [24] the authors proved that only at a certain time scale are the two integer-order oscillators synchronized with each other. We set the time scale $a = 5$ for the wavelet function, and set $\omega_{1,2} = 1.03$ and 0.97 for the two oscillators. For an integer order chaotic system, it was mentioned in Ref. [25] that there will be no phase synchronization if the central frequency Ω_0 is lower than a certain critical value. In Ref. [26], the author adopted an interesting method to investigate the influence of the central frequency Ω_0 on the phase synchronization. Now, we study the impact of the central frequency Ω_0 on phase synchronization in the fractional-order chaotic system. We set the coupling parameter $\varepsilon = 0.5$. The simulation result is shown in Fig. 2. We can see that the phase difference increases with time when $\Omega_0 = 0.5\pi$, π and remains bounded if $\Omega_0 > 1.5\pi$. Thus, in the fractional-order chaotic system, the central frequency Ω_0 is also important for determining the phase synchronization. With the above parameter setting, the phase synchronization threshold lies

somewhere between π and 1.5π.

In the following, we set the central frequency $\Omega_0 = 2\pi$ for the wavelet function to study the effect of coupling strength on phase synchronization. We still set the time scale $a = 5$. The simulation results are shown in Fig. 3. As we can see from Fig. 3(a), when we change the value of the coupling parameter ε, the phase difference decreases with the increasing of the coupling parameter ε. When $\varepsilon = 0.05$ and 0.1, the phase difference increases with time. When $\varepsilon = 0.5$, the phase difference does not increase with time at all and $|\varphi_1(a,t) - \varphi_2(a,t)| < \text{const}$, thus the phase synchronization is achieved for this time scale. In the last case, the amplitudes of W_1 and W_2 remain chaotic and their correlation are pretty small as shown in Fig. 3(b), although the phases are locked.

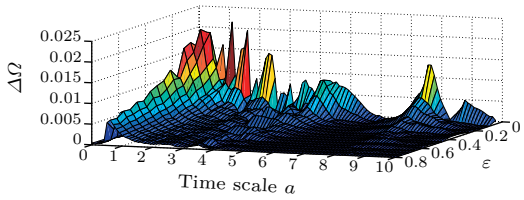


Fig. 4. The mean frequency differences of the coupled fractional-order Rössler equations.

In the above simulation, we show that for a specific time scale, if the central frequency Ω_0 is above a certain critical value, the wavelet phase synchronization can be achieved. Next, we systematically investigate the dependence of the phase synchronization on different coupling strength ε and different time scale a. The coupling parameter ranges from 0 to 0.8, and the time scale a ranges from 0 to 10. The simulation results are shown in Fig. 4. We find that with the increasing coupling strength ε, the mean frequency difference $\Delta\Omega$ for the coupled systems decreases at all time scales, but approaches zero only at certain time scales.

In summary, we have built a coupled fractional-order Rössler system and investigated the phase synchronization behaviors with the approach of complex Morlet-wavelet transform. For other factional-order chaotic systems, as long as we can define the phase appropriately, we can also use the above-mentioned method to investigate the phase synchronization. The study is useful for enhancing our understanding of the phase synchronization of fractional-order chaotic systems. Developing some analysis methods for fractional-order systems and synchronization in fractional-order systems is an interesting topic for future research.

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