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# Generalized Projective Synchronization of Fractional Order Chaotic Systems with Different Dimensions *

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*The generalized projective synchronization of different dimensional fractional order chaotic systems is investigated. According to the stability theory of linear fractional order systems, a sufficient condition to realize synchronization is obtained. The fractional order chaotic and hyperchaotic systems are applied to achieve synchronization in both reduced and increased dimensions. The corresponding numerical results coincide with theoretical analysis.*

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In recent years, chaos synchronization[1,2] of fractional order systems has attracted a great deal of attention due to its great number of potential applications in secure communication and control processing. Various kinds of methods for the synchronization of fractional order chaotic systems are proposed, such as sliding mode control,[3] activation feedback control,[4] and the Laplace method.[5] Projective synchronization (PS) was first reported by Mainieri and Rehacek.[6] Then the PS of integer-order chaotic systems has been widely researched because it can obtain faster communication with its proportional feature.[7–9] However, the PS for fractional order chaotic systems is still a novel topic. In Ref. [10], Wang and Chen proposed a new approach for the PS of a fractional order chaotic system. Peng *et al.* put forward a generalized projective synchronization for two special classes of fractional order chaotic systems.[11,12] Afterwards, Chen *et al.*[13] discussed a lag projective synchronization for fractional order chaotic (hyperchaotic) systems. Moreover, Zhou and Zhu[14] advised a function projective synchronization for fractional order chaotic systems with the same dimensions.

Chaos synchronization between different dimensional systems usually occurs in biological and social sciences.[15,16] However, most of the existing results concentrate on the synchronization of different dimensional integer order chaotic systems.[17,18] Only a few conclusions focus on the chaos synchronization of fractional order systems with different dimensions. In Ref. [19], Wang *et al.* advised a new method for hybrid projective synchronization (HPS) of fractional order chaotic systems with different dimensions. The HPS means that drive and response systems synchronize up to a real scaling matrix. In this Letter, we investigate the generalized projective synchronization (GPS) of different dimensional fractional order chaotic systems. The drive and response systems synchronize up to a continuous scaling function matrix.

The fractional order drive and response systems, respectively, are expressed by

$$D^{\alpha}x = F(x), \tag{1}$$

$$D^{\beta}y = G(y) + U(x, y), \tag{2}$$

where $\alpha, \beta \in (0, 1]$ are the orders of fractional derivatives, $x = (x_1, x_2, \ldots, x_m)^T \in R^m$, $y = (y_1, y_2, \ldots, y_n)^T \in R^n$ are state vectors, $m, n \in N_+$ are system dimensions, $n \neq m$, $F : R^m \to R^m$, $G : R^n \to R^n$ are continuous vector functions and $U : R^{m+n} \to R^n$ is a controller to be designed later.

*Definition*: For the above drive system (1) and response system (2), if there exists a controller $U(x, y)$ and a polynomial matrix $K(x) \in R^{n\times m}$ such that

$$\lim_{t\to+\infty} \|e\| = \lim_{t\to+\infty} \|y - K(x)x\| = 0, \tag{3}$$

then we regard that $x$ synchronizes to $y$ up to a scaling function matrix $K(x)$, and we call such a synchronization the generalized projective synchronization. $e = (e_1, e_2, \ldots, e_n)^T \in R^n$ is an error state vector between systems (1) and (2) for GPS.

*Remark 1*: In particular, when the system dimensions $n < m$, the two systems (1) and (2) are said to be "reduced dimension GPS" if there exists a polynomial matrix

$$K(x) = \begin{pmatrix} k_1(x) & & & & \\ & k_2(x) & & & \\ & & \ddots & & \\ & & & k_n(x) & \cdots \quad k_m(x) \end{pmatrix}$$

such that $\lim_{t\to+\infty} \|e\| = 0$, where $e_i = y_i - k_i(x)x_i, i = 1, 2, \ldots, n-1$ and $e_n = y_n - \sum_{j=n}^{m} k_j(x)x_j$.

*Remark 2*: Similarly, when the system dimensions $n > m$, the two systems (1) and (2) are said to be "increased dimension GPS" if there exists a polynomial

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matrix

$$K(x) = \begin{pmatrix} k_1(x) & & & & \\ & k_2(x) & & & \\ & & \ddots & & \\ & & & k_m(x) & \\ & & & \vdots & \\ & & & k_n(x) \end{pmatrix}$$

such that $\lim_{t\to+\infty} \|e\| = 0$, where $e_i = y_i - k_i(x)x_i$, $i = 1, 2, \ldots, m-1$ and $e_i = y_i - k_i(x)x_m$, $i = m, m+1, \ldots, n$.

With the above parameters, it is assumed that the function

$$U(x, y) = D^\beta(K(x)x) - G(K(x)x) + C(x, y)e, \quad (4)$$

is a suitable controller, in which $C(x, y) \in R^{n\times n}$ is a polynomial matrix to be determined later. Substituting the controller (4) into the response system (2), the error system can be obtained as

$$D^\beta e = (B(x, y) + C(x, y))e, \quad (5)$$

where $B(x, y)e = G(y) - G(K(x)x)$, $B(x, y) \in R^{n\times n}$ is also a polynomial matrix. Then, the GPS between systems (1) and (2) is transformed into the analysis of the asymptotical stability of zero solution of system (5).

*Proposition*: Given the fractional order drive system (1) and response system (2), there exists a polynomial matrix $C(x, y) \in R^{n\times n}$ such that the GPS between systems (1) and (2) can be achieved if

$$[B(x, y) + C(x, y)]M + M[B(x, y) + C(x, y)]^T = -N, \quad (6)$$

where $M$, $N \in R^{n\times n}$ are real symmetric positive definite matrixes.

*Proof*: Presume that $\lambda$ is one of the eigenvalues of matrix $B(x, y) + C(x, y)$ and the corresponding nonzero left eigenvector is $\eta$, i.e.,

$$\eta^T[B(x, y) + C(x, y)] = \lambda\eta^T. \quad (7)$$

Assume that $\lambda^* = \bar{\lambda}$ and $\eta^* = \bar{\eta}$. Here $\lambda^*$ is also the eigenvalue of matrix $B(x, y) + C(x, y)$, i.e.,

$$[B(x, y) + C(x, y)]^T\eta^* = \lambda^*\eta^*, \quad (8)$$

where $B(x, y)$, $C(x, y) \in R^{n\times n}$. Multiply Eq. (6) left by $\eta^T$ and right by $\eta^*$, we can derive that

$$\begin{aligned} &\eta^T[B(x, y) + C(x, y)]M\eta^* \\ &+ \eta^T M[B(x, y) + C(x, y)]^T\eta^* = -\eta^T N\eta^*. \end{aligned} \quad (9)$$

Substituting Eqs. (7) and (8) into Eq. (9), the above equation can be rewritten as

$$\lambda + \lambda^* = -\eta^T N\eta^*/\eta^T M\eta^*. \quad (10)$$

Since $M$ and $N$ are real symmetric positive definite matrixes, one can reach $\lambda + \lambda^* = 2Re(\lambda) < 0$. Hence, for the arbitrary eigenvalue $\lambda$ of matrix $B(x, y) + C(x, y)$, we obtain $|arg(\lambda)| > \pi/2 > \beta\pi/2$. According to the stability theorem of linear fractional order autonomous systems,[2] the equilibrium point of the error system (5) is asymptotically stable. That is, $\lim_{t\to+\infty} \|e\| = 0$, which indicates the two different dimensional fractional order systems (1) and (2) realize the GPS. The proof is completed.

To manifest the effectiveness of the proposed method for GPS, the increased dimension GPS between the fractional order chaotic unified system and the hyperchaotic Chen system is discussed.

Consider the fractional order chaotic unified system

$$\begin{aligned} D^\alpha x_1 &= (25\delta + 10)(x_2 - x_1), \\ D^\alpha x_2 &= (28 - 35\delta)x_1 - x_1x_3 + (29\delta - 1)x_2, \\ D^\alpha x_3 &= x_1x_2 - (\delta + 8)x_3/3, \end{aligned} \quad (11)$$

as the drive system, where $x = (x_1, x_2, x_3)^T$ is a three-dimensional state vector, the parameter $\delta$ always changes in $(0, 1]$. In Ref. [20], the authors found that the lowest order of system (11) to be chaotic is $\alpha = 0.92$.

Choose the fractional order hyperchaotic Chen system

$$\begin{aligned} D^\beta y_1 &= a(y_2 - y_1) + y_4 + u_1, \\ D^\beta y_2 &= by_1 - y_1y_3 + cy_2 + u_2, \\ D^\beta y_3 &= y_1y_2 - dy_3 + u_3, \\ D^\beta y_4 &= y_2y_3 + \gamma y_4 + u_4, \end{aligned} \quad (12)$$

as the response system, where $y = (y_1, y_2, y_3, y_4)^T$ is a 4-dimension state vector, $a$, $b$, $c$, $d$, $\gamma$ are positive parameters, $u = (u_1, u_2, u_3, u_4)^T$ is a controller to be determined later. In Ref. [21], Hegazi and Matouk showed that the lowest order of system (12) to be hyperchaotic is $\beta = 0.94$.

The error state vector is defined as $e = y - K(x)x$, where the matrix $K(x) \in R^{4\times 3}$ is described by

$$K(x) = \begin{pmatrix} k_1(x) & & \\ & k_2(x) & \\ & & k_3(x) \\ & & k_4(x) \end{pmatrix}.$$

With the suitable controller (4), the error system can be written as $D^\beta e = (B(x, y) + C(x, y))e$, where

$$B(x, y) = \begin{pmatrix} -a & a & 0 & 1 \\ b - k_3(x)x_3 & c & -y_1 & 0 \\ y_2 & k_1(x)x_1 & -d & 0 \\ 0 & k_3(x)x_3 & y_2 & \gamma \end{pmatrix}.$$

Selecting the polynomial matrix

$$C(x,y)=\begin{pmatrix} 0 & -a+k_3(x)x_3 & -y_2 & 0 \\ -b & -d_1-c & -k_1(x)x_1 & 0 \\ 0 & y_1 & 0 & -y_2 \\ -1 & -k_3(x)x_3 & 0 & -d_2-\gamma \end{pmatrix},$$

where $d_1$, $d_2>0$ and the real symmetric positive definite matrix $M=\mathrm{diag}\{1,\,1,\,1,\,1\}$, we can obtain

$$[B(x,y)+C(x,y)]M+M[B(x,y)+C(x,y)]^T = \mathrm{diag}\{-2a,\,-2d_1,\,-2d,\,-2d_2\}.$$

Choosing the real symmetric positive definite matrix $N=\mathrm{diag}\{2a,\,2d_1,\,2d,\,2d_2\}$, we can yield $[B(x,y)+C(x,y)]M+M[B(x,y)+C(x,y)]^T=-N$. Afterwards, the fractional order chaotic Unified system (11) and hyperchaotic Chen system (12) achieve the increased dimension GPS based on the above proposition.

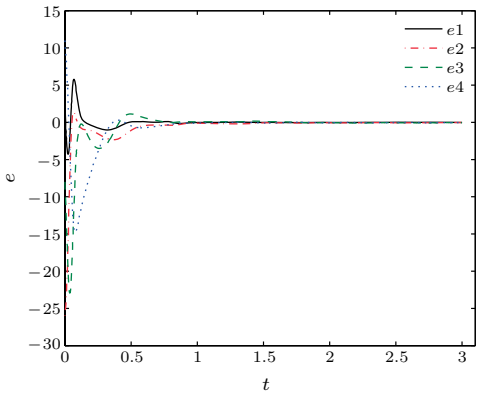


**Fig. 1.** The error state curves of the increased dimension GPS between systems (11) and (12).

For example, when $\alpha=0.92$, $\beta=0.95$, $\delta=0.8$ and $(a,\,b,\,c,\,d,\,\gamma)=(35,\,7,\,12,\,3,\,0.3)$, the drive and response systems (11) and (12) are chaotic. Setting $d_1=3$ and $d_2=5$, the increased dimension GPS between systems (11) and (12) can be realized with a scaling function matrix

$$K(x)=\begin{pmatrix} x_2/100 & & \\ & x_3/3-4 & \\ & & x_1/25+0.4 \\ & & (x_2-x_1)/10 \end{pmatrix}.$$

The corresponding error state curves are displayed in Fig. 1, which indicate that the increased dimension GPS for the fractional order systems (11) and (12) is satisfactorily realized.

In summary, the GPS for two different dimensional fractional order chaotic systems is studied. A nonlinear controller is given to control the slave system to become a projection of the master system with respect to a scaling function matrix. Based on the stability theorem of linear fractional order systems, a synchronization scheme is proposed for the reduced dimension GPS and increased dimension GPS. Finally, the fractional order chaotic unified system and hyperchaotic Chen system are applied to achieve the increased dimension GPS.

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