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Quantum Stackelberg Duopoly with Continuous Distributed Incomplete Information

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A general model of the quantum Stackelberg duopoly is constructed by introducing the "minimal" quantum structure into the Stackelberg duopoly with continuous distributed incomplete information, where both players only know the continuous distribution of the competitor's unit cost. In this model, the cases with complete information, discrete distributed incomplete information, and continuous distributed asymmetric information are all involved. Because of different roles played by the total information uncertainty and the information asymmetry, the game exhibits some new interesting features, such as the total information uncertainty can counteract or improve the first-mover advantage according to the value of the quantum entanglement. What's more, this general model will be helpful for the government to reduce the abuses of oligopolistic competition and to improve the economic efficiency.

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In the past few years, quantum game theory has emerged as a new active field since the concept of quantum information was introduced into the classical game theory. The seminal work by Meyer investigated the coin tossing game, in which a player can always beat his classical opponent by utilizing quantum strategy.[1] Consequently, Eisert *et al.*[2] presented the quantization of the famous prisoners' dilemma and eliminated the dilemma existing in the classical game by quantum strategies. Since then, many quantum structures, such as the "minimal" quantum structure,[3−10] multiplayer quantum structure,[11,12] and asymmetric version,[13] were proposed to investigate the different models in game theory.[14−17] Many interesting and surprising properties of quantum games were obtained theoretically.[18−23] What's more, an economic regulation approach based on the quantum entanglement was also proposed.[24]

It is well known that there exist two distinct kinds of games, i.e. the static and dynamic games.[25] In static games all players make their strategic moves at same time and thus have to guess the action of their opponents, for example, the Cournot and Bertrand duopolies, while in dynamic games all players make their strategic moves at different times. According to the information obtained by the players, each model in game theory has two different conditions, i.e., one with complete information (C-games) and the other with incomplete information (I-games). In C-games all players have full information, while in I-games some players have partial information.

In 2002, Li *et al.*[3] investigated the quantization of games with continuous strategic space instead of discrete strategies, and gave a minimal quantum structure of the Cournot duopoly with complete information. The results show that the dilemma-like situation can disappear by quantizing the game. After that, the Bertrand duopoly with complete information, the quantum Cournot and Bertrand duopolies of continuous distributed incomplete information have been investigated.[4−7] In order to describe the incomplete information, two parameters have been introduced, i.e. the information asymmetry and total information uncertainty, which are all crucial factors that determine the properties of a quantum oligopoly.

As we know, the Stackelberg duopoly is commonly regarded as the simplest dynamic game,[25] where one of the two firms (firm 1) is allowed to move first. The competitor (firm 2) can observe its opponent's move before making its own decision. In such a dynamic game there exists an interesting result, the first-mover advantage. Lo and Kiang applied the quantum scheme of Li *et al.*[3] to the Stackelberg duopoly with complete information,[8] and obtained that positive quantum entanglement would enhance the first-mover advantage beyond the classical limit. Furthermore, the Stackelberg duopoly with discrete distributed incomplete information and continuous distributed asymmetric information were all investigated the same way,[9,10] which indicated that the first-mover advantage is suppressed by the incomplete information although positive quantum entanglement improves the first-mover tolerance for the incomplete information. After that, the influences of decoherence, acceleration, correlated noise, and damping factor to the quantum Stackelberg duopoly game under certain conditions were all investigated.[18−21]

In this Letter, we use the minimal quantization

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rules of Li *et al.*[3] to investigate the quantum version of Stackelberg duopoly with continuous distribution incomplete information, where both players only know the continuous distribution of the competitor's unit cost. Following the studies in Refs. [6,7,10], we also introduce two parameters, i.e. the information asymmetry and total information uncertainty, which will play important roles in the model and result in some new interesting features.

To begin with, we review the classical Stackelberg duopoly with continuous distributed incomplete information, in which two firms, i.e. firm 1 and firm 2, produce a homogeneous product for the market. Firm i only knows the continuous distribution $f_j(c_j)$ of its opponent's unit cost c_j $(i,j = 1,2)$. Accordingly, two firms' profits can be expressed as

$$\begin{aligned}\mu_1(q_1,q_2) &= \int f_2(c_2)q_1(a-c_1-q_1-q_2)dc_2,\\ \mu_2(q_1,q_2) &= \int f_1(c_1)q_2(a-c_2-q_1-q_2)dc_1,\end{aligned} \tag{1}$$

where q_i is the product quantity of firm i, and a is a market constant which satisfies $a > \max\{c_1, c_2\}$.

In the following, we apply the minimal quantization rules of Li *et al.*[3] to the quantum extension of the Stackelberg duopoly with continuous distributed incomplete information. In the quantization scheme,[3] the game starts from the state $|\text{vac}\rangle_1 \otimes |vac\rangle_2$, which is the direct product of two single-mode electromagnetic fields. This state undergoes a unitary entanglement operation $\hat{J}(\gamma) \equiv \exp\{-\gamma(\hat{a}_1^+\hat{a}_2^+ - \hat{a}_1\hat{a}_2)\}$, both firms and symmetry with respect to interchange of the two field modes guarantee a fair competition. Here $\gamma \geq 0$ is known as the squeezing parameter and can be regarded as a measure of quantum entanglement, and the operators $\hat{a}_1$ $(\hat{a}_1^+)$ and $\hat{a}_2$ $(\hat{a}_2^+)$ are the annihilation (creation) operators of the electromagnetic fields. Strategic movements of the two firms are associated with the unitary operations $\hat{D}_1(x_1) = \exp\{x_1(\hat{a}_1^+ - \hat{a}_1)/\sqrt{2}\}$ $(x_1 \in [0,+\infty))$ and $\hat{D}_2(x_2) = \exp\{x_2(\hat{a}_2^+ - \hat{a}_2)/\sqrt{2}\}$ $(x_2 \in [0,+\infty))$, respectively. Finally, both players' states are measured after a disentanglement operation $\hat{J}^+(\gamma)$.[3] Therefore the final state prior to the measurement is equal to $|\psi_f\rangle \equiv \hat{J}(\gamma)^+\hat{D}_1(x_1)\hat{D}_2(x_2)\hat{J}(\gamma)|\text{vac}\rangle_1 \otimes |\text{vac}\rangle_2$, which is a product of two coherent states $|q_1\rangle_1$ and $|q_2\rangle_2$. In order to reduce the noise of the measurement result, the final measurement was carried out by the homodyne measurement with an infinitely squeezed reference light beam.[3] The measured results are the eigenvalues of the final coherent states and represent the quantities of two firms denoted by

$$\begin{aligned}q_1 &= x_1\cosh\gamma + x_2\sinh\gamma,\\ q_2 &= x_1\sinh\gamma + x_2\cosh\gamma.\end{aligned} \tag{2}$$

Substituting Eq. (2) into Eq. (1), the two firms' profit functions can be given by

$$\begin{aligned}\mu_1(q_1,q_2) &= \int f_2(c_2)(x_1\cosh\gamma + x_2\sinh\gamma)\\ &\quad\cdot(a-c_1-x_1e^{\gamma}-x_2e^{\gamma})dc_2,\\ \mu_2(q_1,q_2) &= \int f_1(c_1)(x_1\sinh\gamma + x_2\cosh\gamma)\\ &\quad\cdot(a-c_2-x_2e^{\gamma}-x_1e^{\gamma})dc_1.\end{aligned} \tag{3}$$

The solution to this duopoly problem can be obtained by the methods of subgame perfect equilibrium and backwards induction. This means that we first solve for the second mover (firm 2). Firm 2 reacts optimally to q_1 by choosing q_2 which maximizes its profit. Then firm 1 takes into account the optimal response of firm 2 and sets its own move in such a way to maximize its own profit. That is, firm 1 substitutes reaction function of firm 2 into its own profit function and then optimizes it by setting the first derivative of the expected profit function with respect to q_1 that equal to zero. Consequently, the solution is given by

$$\begin{aligned}x_1^* &= -\frac{1+\cosh\gamma e^{-\gamma}}{e^{\gamma}+2\cosh\gamma}c_1 + \frac{\cosh^2\gamma e^{-\gamma}}{1+\cosh\gamma e^{-\gamma}}a\\ &\quad+\frac{1-2\sinh\gamma\cosh\gamma}{(1+\cosh\gamma e^{-\gamma})(e^{\gamma}+2\cosh\gamma)}c,\\ x_2^* &= -\frac{1}{2}e^{-\gamma}c_2 + \frac{1}{2}e^{-\gamma}\Big(1-\frac{\cosh\gamma e^{\gamma}}{1+\cosh\gamma e^{-\gamma}}\Big)a\\ &\quad+\frac{\cosh\gamma}{2(1+\cosh\gamma e^{-\gamma})}c.\end{aligned} \tag{4}$$

For simplicity we have assumed $\overline{c_2} = \overline{c_1} = c$, where $\overline{c_i}$ $(i = 1,2)$ is the averaged value of unit cost c_i.

To avoid the complexity caused by specific choice of firm i's unit cost c_i, we shall average over c_i and have the average optimal profits,

$$\begin{aligned}\overline{\mu_1} &= -\frac{1}{4}e^{-\gamma}\sinh\gamma\overline{\Delta c_2^2} + \frac{\cosh^2\gamma(1+\cosh\gamma e^{-\gamma})}{(e^{\gamma}+2\cosh\gamma)^2}\overline{\Delta c_1^2}\\ &\quad+\frac{1}{4(1+\cosh\gamma e^{-\gamma})}k^2,\\ \overline{\mu_2} &= \frac{1}{4}e^{-\gamma}\cosh\gamma\overline{\Delta c_2^2} - \frac{e^{\gamma}\sinh\gamma(1+\cosh\gamma e^{-\gamma})^2}{(e^{\gamma}+2\cosh\gamma)^2}\overline{\Delta c_1^2}\\ &\quad+\frac{\cosh\gamma e^{-\gamma}}{4(1+\cosh\gamma e^{-\gamma})^2}k^2,\end{aligned} \tag{5}$$

where $k = a - c$, $\overline{\Delta c_i^2} = \int f_i(c_i)(c_i - \overline{c_i})^2dc_i$ is the information fluctuation of c_i. As a result, the difference between the expected profits of two firms can be evaluated as

$$\begin{aligned}\delta_{\text{quantum}} &= \overline{\mu_1} - \overline{\mu_2}\\ &= -\frac{1}{4}\overline{\Delta c_2^2} + \frac{e^{2\gamma}(1+\cosh\gamma e^{-\gamma})}{(e^{\gamma}+2\cosh\gamma)^2}\overline{\Delta c_1^2}\\ &\quad+\frac{1}{4(1+\cosh\gamma e^{-\gamma})^2}k^2,\end{aligned} \tag{6}$$

where $\overline{\mu_1}$ and $\overline{\mu_2}$ are average optimal profits and δ_{quantum} represents the first-mover advantage when $\delta_{\text{quantum}} \geq 0$. From Eqs. (5) and (6), one can see that some terms including parameters $\overline{\Delta c_1^2}$ and $\overline{\Delta c_2^2}$ are introduced into the equations, which enable us to summarize the models of complete information ($\overline{\Delta c_2^2} = \overline{\Delta c_1^2} = 0$),[8] discrete distribution incomplete information, and continuous distribution asymmetric information together.[9,10]

In order to describe the relations of the two players with respect to information, we have an alternate representation of the incomplete information by defining

$$M = \frac{\overline{\Delta c_1^2} - \overline{\Delta c_2^2}}{k^2}, \quad N = \frac{\overline{\Delta c_1^2} + \overline{\Delta c_2^2}}{k^2}, \tag{7}$$

where M and N represent the information asymmetry and the total information uncertainty, respectively.[6,7,11] Then we have

$$\begin{aligned}
\overline{\mu_1} &= \frac{k^2}{8}\left[e^{-2\gamma} - \frac{1+\sinh\gamma e^{\gamma}}{(e^{\gamma}+2\cosh\gamma)^2}\right]N \\
&\quad + \frac{k^2}{8}\left[1 - \frac{\cosh\gamma e^{\gamma}}{(e^{\gamma}+2\cosh\gamma)^2}\right]M \\
&\quad + \frac{1}{4(1+\cosh\gamma e^{-\gamma})}k^2 \\
&\equiv a_1(\gamma)N + a_2(\gamma)M + \frac{1}{4(1+\cosh\gamma e^{-\gamma})}k^2, \\
\overline{\mu_2} &= \frac{k^2}{8}\left[\cosh\gamma e^{-\gamma} - \frac{4\sinh\gamma e^{\gamma}(1+\cosh\gamma e^{-\gamma})^2}{(e^{\gamma}+2\cosh\gamma)^2}\right]N \\
&\quad + \frac{k^2}{8}\left[-\cosh\gamma e^{-\gamma} - \frac{4\sinh\gamma e^{\gamma}(1+\cosh\gamma e^{-\gamma})^2}{(e^{\gamma}+2\cosh\gamma)^2}\right] \\
&\quad \cdot M + \frac{\cosh\gamma e^{-\gamma}}{4(1+\cosh\gamma e^{-\gamma})^2}k^2 \\
&\equiv a_3(\gamma)N + a_4(\gamma)M + \frac{\cosh\gamma e^{-\gamma}}{4(1+\cosh\gamma e^{-\gamma})^2}k^2,
\end{aligned} \tag{8}$$

$$\begin{aligned}
\delta_{\text{quantum}} &= \frac{k^2}{8}\frac{3e^{2\gamma} - 4\cosh^2\gamma}{(e^{\gamma}+2\cosh\gamma)^2}N \\
&\quad + \frac{k^2}{8}\left[1 + \frac{4e^{2\gamma}+4\cosh\gamma e^{\gamma}}{(e^{\gamma}+2\cosh\gamma)^2}\right]M \\
&\quad + \frac{1}{4(1+\cosh\gamma e^{-\gamma})^2}k^2 \\
&\equiv a_5(\gamma)N + a_6(\gamma)M + \frac{1}{4(1+\cosh\gamma e^{-\gamma})^2}k^2.
\end{aligned} \tag{9}$$

From Eqs. (8) and (9), one can see that $\overline{\mu_1}$, $\overline{\mu_2}$ and δ_{quantum} are linear functions of the variables N and M, $a_i(\gamma)\,(i = 1, \ldots, 6)$ are the corresponding linear coefficients.

Next, we discuss the influence of variables N, M and γ on $\overline{\mu_1}$, $\overline{\mu_2}$ and δ_{quantum}, respectively. To analyze the main characteristics we focus on two limiting cases. Firstly, we consider the case of $M = 0$ and $N \neq 0$, where the effect of incomplete information is purely attributed to the total information uncertainty N. From Eq. (8) it follows that $a_1(\gamma_1) = 0$ and $a_3(\gamma_2) = 0$ give two special values γ_1 and γ_2, and $0 < \gamma_2 < \gamma_1$. When the quantum entanglement $\gamma \geq \gamma_1$ the two firms' profits $\overline{\mu_1}$ and $\overline{\mu_2}$ decrease with the increasing N for the reasons of $a_1(\gamma) < 0$ and $a_3(\gamma) < 0$ when $\gamma_2 \leq \gamma < \gamma_1$, $a_1(\gamma) > 0$ and $a_3(\gamma) < 0$, $\overline{\mu_1}$ increases but $\overline{\mu_2}$ decreases with the increasing of N, when $0 \leq \gamma < \gamma_2$, $a_1(\gamma) > 0$ and $a_3(\gamma) > 0$, then $\overline{\mu_1}$ and $\overline{\mu_2}$ increase with the increasing N. In the same way, $a_5(\gamma_3) = 0$ gives a special value of γ_3. The difference between the expected profits δ_{quantum} increases with increasing N for $\gamma \geq \gamma_3$ due to $a_5(\gamma) > 0$, while decreases with increasing N for $0 \leq \gamma < \gamma_3$ due to $a_5(\gamma) < 0$. These mean that the effects of N on $\overline{\mu_1}$, $\overline{\mu_2}$, δ_{quantum} and the first-mover advantage are dramatically influenced by γ. Figure 1 shows $\overline{\mu_1}$, $\overline{\mu_2}$ and δ_{quantum} versus N and γ with $M = 0$, in which all features can be seen.

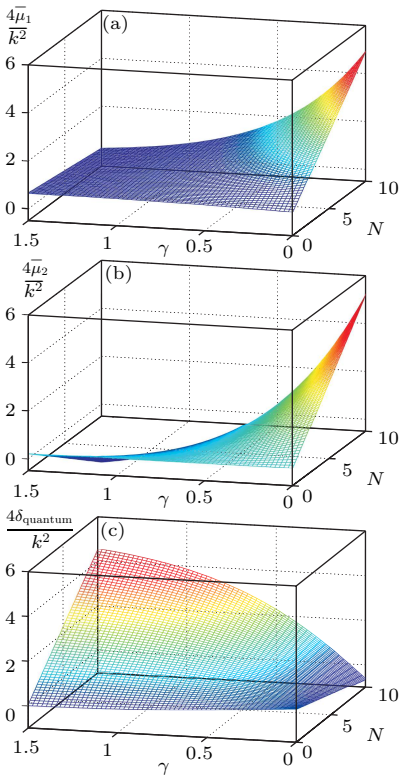


Fig. 1. Behavior of (a) $\overline{\mu_1}$, (b) $\overline{\mu_2}$ and (c) δ_{quantum} as functions of N and γ with $M = 0$.

Secondly, we consider the case of $N = 0$ and $M \neq 0$, where the effect of incomplete information is purely attributed to the information asymmetry M. As shown in Eqs. (8) and (9), $a_2(\gamma) > 0$, $a_4(\gamma) < 0$ and $a_6(\gamma) > 0$ at any γ. It is indicated that increasing M will reduce $\overline{\mu_2}$ and enhance $\overline{\mu_1}$ and δ_{quantum}. What's more, for $N = 0$ δ_{quantum} is positive, which denotes that M will gradually enhance the first-mover advantage. Figure 2 shows the $\overline{\mu_1}$, $\overline{\mu_2}$ and δ_{quantum} as

functions of M and γ with $N = 0$, where all features can also be seen.

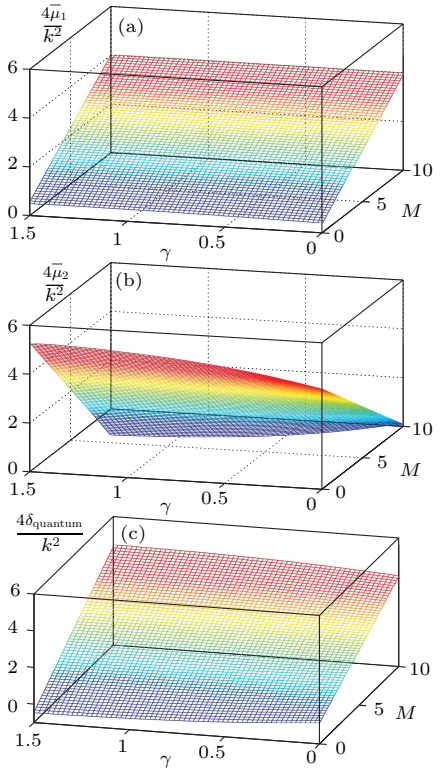


Fig. 2. (a) Behavior of (a) $\overline{\mu_1}$, (b) $\overline{\mu_2}$ and (c) δ_{quantum} as functions of M and γ with $N = 0$.

Finally, we investigate the effect of γ on $\overline{\mu_1}$, $\overline{\mu_2}$ and δ_{quantum}, which is dramatically influenced by M and N. From Fig. 1(a), it can be seen that for $M = 0$, there is a threshold in N when $N = N_1$. For $0 \leq N < N_1$, $\overline{\mu_1}$ increases with increasing γ, while for $N \geq N_1$, $\overline{\mu_1}$ decreases with increasing γ. It is indicated that the effect of γ on $\overline{\mu_1}$ is influenced by N, but almost is not influenced by M (see Fig. 2(a)).

From Figs. 1(b) and 2(b), it can be seen that the effect of γ on $\overline{\mu_2}$ will vary with N and M. Figure 1(b) shows that for $M = 0$, γ will gradually reduce $\overline{\mu_2}$ at any N. What's more, the larger the value of N, the greater the effect of γ on $\overline{\mu_2}$. As shown in Fig. 2(b), for $N = 0$, there exists a threshold in M when $M = M_1$. For $0 \leq M < M_1$, $\overline{\mu_2}$ increases with increasing γ but for $M \geq M_1$, $\overline{\mu_2}$ decreases with increasing γ.

On the other hand, the effect of γ on δ_{quantum} is also influenced by N and M. As shown in Fig. 1(c), for $M = 0$, there exists a threshold in N when $N = N_2$, δ_{quantum} decreases with increasing γ for $0 \leq N < N_2$ but increases with increasing γ for $N \geq N_2$. Figure 2(c) shows that for $N = 0$, there exists a threshold in M when $M = M_2$. δ_{quantum} decreases with increasing γ for $0 \leq M < M_2$ but increases with increasing γ for $M \geq M_2$

In summary, a general model of the quantum version of the Stackelberg duopoly is constructed, where each case of complete information, discrete distributed incomplete information, and continuous distributed asymmetric information can be reproduced by setting special parameters. In this model, both players only know the continuous distribution of its competitor's unit cost. The information asymmetry will gradually enhance the first-mover advantage, and the information uncertainty will counteract or improve the first-mover advantage according to the value of the quantum entanglement. In contrast with the case of complete information where positive entanglement enhances the first-mover advantage beyond the classical limit, the effect of the quantum entanglement on the outcomes, the difference between the expected profits of the two players and the first-mover advantage are more complicated due to the total information uncertainty and the information asymmetry. What's more, this general model will be helpful for the government to reduce the abuses of oligopolistic competition and to improve the economic efficiency.

References

[1] D A Meyer 1999 *Phys. Rev. Lett.* **82** 1052
[2] Eisert J, Wilkens M and Lewenstein M 1999 *Phys. Rev. Lett.* **83** 3077
[3] Li H, Du J F and Massar S 2002 *Phys. Lett.* A **306** 73
[4] Lo C F and Kiang D 2004 *Phys. Lett.* A **321** 94
[5] Du J F, Li H and Ju C Y 2003 *Phys. Rev.* E **68** 016124
[6] Qin G, Chen X, Sun M et al 2005 *J. Phys. A: Math. Theor.* **38** 4247
[7] Chen X, Qin G, Zhou X Y et al 2005 *Chin. Phys. Lett.* **22** 1033
[8] Lo C F and Kiang D 2003 *Phys. Lett.* A **318** 333
[9] Lo C F and Kiang D 2005 *Phys. Lett.* A **346** 65
[10] Wang X, Yang X H, Miao L et al 2007 *Chin. Phys. Lett.* **24** 3040
[11] Benjamin S C and Hayden P M 2001 *Phys. Rev.* A **64** 030301
[12] Lo C F and Kiang D 2003 *Europhys. Lett.* **64** 592
[13] Qin G, Chen X, Sun M et al 2005 *Phys. Lett.* A **340** 78
[14] Li Y, Qin G, Zhou X Y et al 2006 *Phys. Lett.* A **355** 447
[15] Khan S, Ramzan M and Khan M K 2010 *Chin. Phys. Lett.* **27** 080302
[16] Li S B 2011 *J. Phys. A: Math. Theor.* **44** 295302
[17] Sekiguchi Y, Sakahara K and Sato T 2010 *J. Phys. A: Math. Theor.* **43** 145303
[18] Zhu X and Kuang L M 2007 *J. Phys. A: Math. Theor.* **40** 7729
[19] Zhu X and Kuang L M 2008 *Commun. Theor. Phys.* **49** 111
[20] Khan S, Ramzan M and Khan M K 2010 *J. Phys. A: Math. Theor.* **43** 375301
[21] Khan S and Khan M K 2011 *Chin. Phys. Lett.* **28** 070202
[22] Guo H, Zhang J and Koehler G J 2008 *Desion Support Syst.* **46** 318
[23] Du J F, Li H, Xu X et al 2002 *Phys. Rev. Lett.* **88** 137902
[24] Du J F, Ju C Y and Li H 2005 *J. Phys. A: Math. Theor.* **38** 1559
[25] Gibbons R 1992 *Game Theory for Applied Economists* (Princeton NJ: Princeton University)
Bierman H S Fernandez L 1998 *Game Theory with Economic Applications* 2nd edn (Reading, MA: Sddison-Wesley)