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Sound Propagation in a Wedge with a Rigid Bottom *

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Sound propagation in a wedge with perfectly reflecting boundaries is one of the few range-dependent problems with an analytical solution. Since sound propagation towards the wedge apex will be completely backscattered due to the perfectly reflecting boundaries, this test problem is an ideal benchmark for a full two-way solution to the wave equation. An analytical solution for sound propagation in a wedge with a pressure-release sea surface and a pressure-release bottom was presented by Buckingham et al. [J. Acoust. Soc. Am. 87 (1990) 1511]. The ideal wedge problem with a rigid bottom is also of great importance in underwater acoustics. We present an analytical solution to the problem with a wedge bounded above by a pressure-release sea surface and below by a rigid bottom, which may be used to provide informative means of investigating the sound field in depth-varying channels, and to establish the accuracy of numerical propagation models for which it is difficult to treat problems with a pressure-release bottom. A comparison of the analytical solution and the numerical solution recently proposed by Luo et al. [Chin. Phys. Lett. 29 (2012) 014302] is also presented, indicating that this numerical propagation model provides high accuracy.

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Modeling sound propagation in range-dependent waveguides is of great interest in underwater acoustics. A variety of approaches[1−9] have been developed for solving this problem over the past several decades, among which the normal mode method proves to be very important. For problems characterized by a weak range-dependence variation, mode coupling is negligible and the adiabatic mode method proposed by Pierce[3] is sufficient. Otherwise, mode coupling is significant and thus cannot be neglected. Noteworthy advances have been achieved by Fawcett,[10] Godin,[11] and Athanassoulis *et al.*[12] In the cited papers, additional terms are introduced in the series of local vertical normal modes to handle the inconsistency of such modes at nonhorizontal interfaces or bottom boundaries. An alternative approach, usually referred to as the stepwise coupled-mode method, is also very important for treating the range-dependent propagation problem. Developed by Evans, COUPLE is a widely used numerical model applying the stepwise coupled-mode method.[13,14]

A stepwise coupled-mode method with the use of the direct global matrix (DGM) approach was recently proposed by Luo *et al.*[15,16] Due to the use of the DGM approach, this method is numerically stable. In addition, the introduction of general source conditions enables this method to handle the scenario where a line source is located right on top of a sloping bottom. Furthermore, closed-form expressions for coupling matrices are given for propagation in a waveguide involving a pressure-release sea surface, a pressure-release bottom, and a homogenous water column.

Analytical solutions to range-dependent problems are extremely useful for establishing the accuracy of numerical range-dependent propagation models. An analytical solution was proposed by Buckingham *et al.*[17] to solve a wedge problem involving a line source parallel to the wedge apex, a planar pressure-release sea surface, a planar pressure-release bottom, and a homogeneous water column supporting the acoustic field. This is one of the few range-dependent problems with an exact analytical solution. Moreover, this problem is an ideal benchmark for a two-way numerical solution to the wave equation, in which sound propagating to the apex will be completely backscattered due to the perfectly reflecting boundaries. However, it is difficult for certain numerical propagation models to handle a pressure-release bottom. Thus, an analytical solution to the wedge problem with a rigid bottom is also of great interest to the underwater acoustics community.

In this Letter, we first present an analytical solution to the ideal wedge problem with a rigid bottom. Second, we give closed-form expressions for the coupling matrices for propagation in a waveguide bounded above by a pressure-release sea surface and below by a rigid bottom. Third, we apply the numerical method presented in Ref. [15] to this wedge problem. A comparison of the analytical and numerical solutions followed by a discussion is also provided.

The geometry of the wedge problem is illustrated in Fig. 1. The acoustic field is supported by homogeneous water, and the line source is parallel to the apex. A pressure-release sea surface and a rigid bottom constitute the boundaries of the wedge. A two-dimensional cylindrical coordinate system with the origin at the apex is adopted in deriving the analytical solution.

The Helmholtz equation describing this problem is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}+k^2 p=-\frac{\delta(r-r_{\rm s})}{r}\delta(\theta-\theta_{\rm s}), \quad (1)$$

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where (r_s, θ_s) denotes the source location and k is the wavenumber in water. The solution is expressed as

$$p(r,\theta) = \sum_{n=1}^{\infty} \Theta_n(\theta) R_n(r). \tag{2}$$

By applying separation of variables, together with the boundary condition at the sea surface, $\Theta_n(0) = 0$, and the boundary condition at the bottom,

$$\frac{d\Theta_n}{d\theta}|_{\theta=\theta_0} = 0 \tag{3}$$

with θ_0 denoting the wedge angle, we reach the following solutions for $\Theta_n(\theta)$ and $R_n(r)$:

$$\Theta_n(\theta) = \sqrt{\frac{2}{\theta_0}} \sin(\gamma_n \theta), \tag{4}$$

$$R_n(r) = \frac{\pi}{2} i J_{\gamma_n}(kr_<) H^{(1)}_{\gamma_n}(kr_>) \Theta_n(\theta_s), \tag{5}$$

where $J_{\gamma_n}(\cdot)$ and $H^{(1)}_{\gamma_n}(\cdot)$ are, respectively, the Bessel and Hankel functions of the first kind of order γ_n, with

$$\gamma_n = \left(n - \frac{1}{2}\right)\frac{\pi}{\theta_0}, \quad n = 1, 2, \ldots \tag{6}$$

$$r_< = \min(r, r_s), \tag{7}$$

$$r_> = \max(r, r_s). \tag{8}$$

Therefore, we obtain the following analytical solution to the Helmholtz equation (1),

$$p(r,\theta) = \frac{i\pi}{\theta_0} \sum_{n=1}^{\infty} \sin(\gamma_n \theta_s) \sin(\gamma_n \theta) J_{\gamma_n}(kr_<) H^{(1)}_{\gamma_n}(kr_>). \tag{9}$$

It is worth addressing that for the wedge problem with a pressure-release bottom, the solution is exactly of the same form as Eq. (9), except that in this case the boundary condition at the bottom, $\Theta_n(\theta_0) = 0$, leads to

$$\gamma_n = n\frac{\pi}{\theta_0}, \quad n = 1, 2, \ldots \tag{10}$$

A numerical model for solving sound propagation in range-dependent waveguides was recently proposed by Luo *et al.*[15,16] In this model, Cartesian coordinates are introduced, with the source location denoted by (x_s, z_s). The Helmholtz equation is

$$\frac{\partial^2 p}{\partial x^2} + \rho(z)\frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\frac{\partial p}{\partial z}\right) + \frac{\omega^2}{c^2(z)} p = -\delta(x - x_s)\delta(z - z_s). \tag{11}$$

We divide the range-dependent waveguide into a number of range-independent segments and express the field solution in segment j as

$$p^j(x,z) = \sum_{n=1}^{N} [a_n^j E_{1n}^j(x) + b_n^j E_{2n}^j(x)] \Psi_n^j(z), \tag{12}$$

where N is the total number of normal modes involved in mode coupling, a_n^j and b_n^j are modal amplitudes corresponding to the forward and backward propagating waves, respectively, $\Psi_n^j(z)$ are the local vertical eigenfunctions, $E_{1n}^j(x)$ and $E_{2n}^j(x)$ are normalized exponential functions defined as follows:

$$E_{1n}^j(x) = \exp[ik_{xn}^j(x - x^{j-1})], \tag{13}$$

$$E_{2n}^j(x) = \exp[-ik_{xn}^j(x - x^j)], \tag{14}$$

with k_{xn}^j denoting the horizontal wavenumbers. The solutions for $\boldsymbol{a}^j$ and $\boldsymbol{b}^j$ in each segment are obtained by solving the following linear system of equations,

$$\boldsymbol{C}\boldsymbol{x} = \boldsymbol{v}, \tag{15}$$

where $\boldsymbol{x}$ is the column vector

$$\boldsymbol{x} = [\,\boldsymbol{b}^1 \quad \boldsymbol{a}^2 \quad \boldsymbol{b}^2 \quad \boldsymbol{a}^3 \cdots \boldsymbol{b}^{j_s} \quad \boldsymbol{a}^{j_s+1} \quad \cdots \quad \boldsymbol{b}^{J-1} \quad \boldsymbol{a}^J\,]^{\mathrm{T}}, \tag{16}$$

$\boldsymbol{v}$ is the column vector

$$\boldsymbol{v} = [\,\boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \boldsymbol{0} \quad \cdots \quad \boldsymbol{0} \quad \boldsymbol{s} \quad \cdots \quad \boldsymbol{0} \quad \boldsymbol{0}\,]^{\mathrm{T}}, \tag{17}$$

with the superscript T denoting transpose, and $\boldsymbol{s}$ being the column vector with entries

$$s_n = -\frac{i}{\rho(z_s)} \frac{\Psi_n^{j_s}(z_s)}{k_{xn}^{j_s}}, \quad n = 1, 2, \ldots, N, \tag{18}$$

and $\boldsymbol{C}$ is the global coefficient matrix

$$\boldsymbol{C} = \begin{bmatrix}
\widetilde{\boldsymbol{C}}^1 & -\boldsymbol{I} & -\boldsymbol{E}_2^2 & & & & & & \\
-\widehat{\boldsymbol{C}}^1 & -\boldsymbol{I} & \boldsymbol{E}_2^2 & & & & & & \\
 & \widetilde{\boldsymbol{C}}^2\boldsymbol{E}_1^2 & \widetilde{\boldsymbol{C}}^2 & -\boldsymbol{I} & -\boldsymbol{E}_2^3 & & & & \\
 & \widehat{\boldsymbol{C}}^2\boldsymbol{E}_1^2 & -\widehat{\boldsymbol{C}}^2 & -\boldsymbol{I} & \boldsymbol{E}_2^3 & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & \widetilde{\boldsymbol{C}}^{j_s}\boldsymbol{E}_1^{j_s} & \widetilde{\boldsymbol{C}}^{j_s} & -\boldsymbol{I} & -\boldsymbol{E}_2^{j_s+1} & & \\
 & & & \widehat{\boldsymbol{C}}^{j_s}\boldsymbol{E}_1^{j_s} & -\widehat{\boldsymbol{C}}^{j_s} & -\boldsymbol{I} & \boldsymbol{E}_2^{j_s+1} & & \\
 & & & & & \ddots & \ddots & \ddots & \ddots \\
 & & & & & & \widetilde{\boldsymbol{C}}^{J-1}\boldsymbol{E}_1^{J-1} & \widetilde{\boldsymbol{C}}^{J-1} & -\boldsymbol{I} \\
 & & & & & & \widehat{\boldsymbol{C}}^{J-1}\boldsymbol{E}_1^{J-1} & -\widehat{\boldsymbol{C}}^{J-1} & -\boldsymbol{I}
\end{bmatrix}. \tag{19}$$

Coupling matrices $\widetilde{C}^j$ and $\widehat{C}^j$ are respectively defined as

$$\widetilde{C}^j_{\nu n} = \int_0^\infty \frac{1}{\rho^{j+1}(z)} \Psi^{j+1}_\nu(z)\, \Psi^j_n(z) dz, \tag{20}$$

$$\widehat{C}^j_{\nu n} = \frac{k^j_{xn}}{k^{j+1}_{x\nu}} \int_0^\infty \frac{1}{\rho^j(z)} \Psi^{j+1}_\nu(z)\, \Psi^j_n(z) dz. \tag{21}$$

For an ideal waveguide involving a pressure-release surface, a rigid bottom, and a homogeneous water column, we have analytical modal solutions.[18] By substituting the analytical modal solutions into Eqs. (20) and (21), we obtain the following closed-form expressions for the coupling matrices. For upslope propagation, we have

$$\widetilde{C}^j_{\nu n} = FI, \tag{22}$$

$$\widehat{C}^j_{\nu n} = \frac{k^j_{xn}}{k^{j+1}_{x\nu}} F\Big[I + (-1)^{\nu-1} \frac{2\cos(k^j_{zn} D^{j+1})}{k^j_{zn}}\Big], \tag{23}$$

and for downslope propagation we have

$$\widetilde{C}^j_{\nu n} = F\Big[I + (-1)^{n-1} \frac{2\cos(k^{j+1}_{z\nu} D^j)}{k^{j+1}_{z\nu}}\Big], \tag{24}$$

$$\widehat{C}^j_{\nu n} = \frac{k^j_{xn}}{k^{j+1}_{x\nu}} FI, \tag{25}$$

where

$$I = \begin{cases} \frac{\sin[(k^{j+1}_{z\nu} - k^j_{zn})D]}{k^{j+1}_{z\nu} - k^j_{zn}} - \frac{\sin[(k^{j+1}_{z\nu} + k^j_{zn})D]}{k^{j+1}_{z\nu} + k^j_{zn}}, & k^{j+1}_{z\nu} \neq k^j_{zn}, \\ D - \frac{\sin[(k^{j+1}_{z\nu} + k^j_{zn})D]}{k^{j+1}_{z\nu} + k^j_{zn}}, & k^{j+1}_{z\nu} = k^j_{zn}, \end{cases} \tag{26}$$

$$F = (D^j D^{j+1})^{-1/2}, \tag{27}$$

$$D = \min(D^j, D^{j+1}), \tag{28}$$

with D^j and D^{j+1} denoting water depths in segments j and $j+1$, respectively.

In this study, the field results are shown in terms of transmission loss (TL) defined as

$$\mathrm{TL} = -20 \log_{10} |\frac{p(\boldsymbol{r})}{p_0(1)}|, \tag{29}$$

where $\boldsymbol{r}$ is the location of a field point, and the reference pressure takes the form

$$p_0(x) = \frac{i}{4} H^{(1)}_0(k_0 x), \tag{30}$$

where k_0 denotes the reference wavenumber, defined as $k_0 = 2\pi f/1500$, with f denoting the source frequency.

Consider the wedge problem as illustrated in Fig. 2. The environment consists of a homogeneous water column with sound speed $c_\mathrm{w} = 1500\,\mathrm{m/s}$ and density $\rho_\mathrm{w} = 1.0\,\mathrm{g/cm^3}$, bounded above by a pressure-release sea surface and below by a rigid bottom. The wedge angle is approximately 2.86°. We set the origin on the sea surface where the water depth is 400 m. The source is located at a range of 4 km and a depth of 100 m, and the source frequency is 25 Hz. Discussion of the numerical solution to this wedge problem involving a pressure-release bottom can be found in Refs. [15,16].

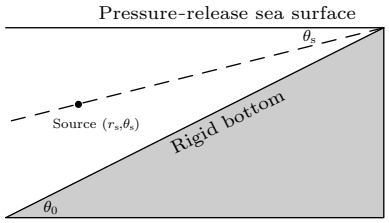

Fig. 1. Geometry of a wedge bounded above by a planar pressure-release sea surface and below by a planar rigid bottom. A line source is involved, which is parallel to the apex and located at $(r_\mathrm{s}, \theta_\mathrm{s})$. The wedge angle is denoted by θ_0.

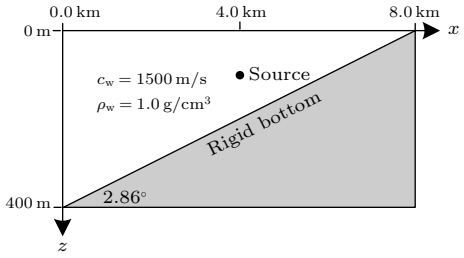

Fig. 2. Geometry of the ideal wedge problem involving a pressure-release sea surface and a rigid bottom.

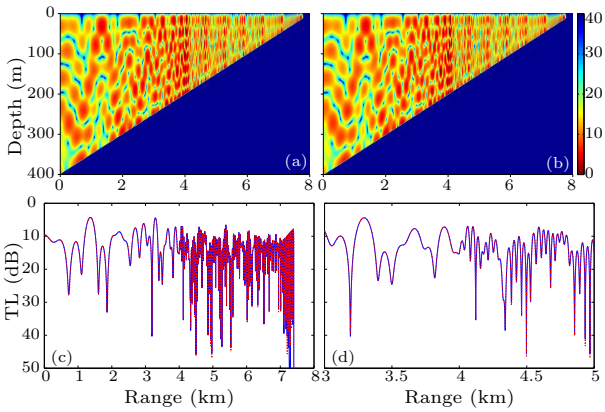

Fig. 3. Field results for the ideal wedge problem with a rigid bottom. A line source parallel to the apex is located at a range of 4 km and a depth of 100 m, and the source frequency is 25 Hz. (a) Analytical result in water; (b) numerical result in water; (c) results at 30-m depth from 0 km to 8 km; (d) results at 30-m depth from 3 km to 5 km. In (c) and (d), the blue solid line is the analytical result, and the red dotted line is the numerical result.

In the implementation of the coupled-mode model, a total of 1200 segments is used to represent the slope, and a total of 30 normal modes is involved in mode coupling. A comparison of the field results by the analytical solution in Eq. (9) and by the numerical propagation model proposed by Luo *et al.*[15,16] is provided in Fig. 3, which shows excellent overall agreement. The positions of the peaks and troughs are the

same, and the absolute levels of the curves are also in good agreement, within a fraction of one dB. Since the ideal wedge problem is a particularly stringent test for numerical propagation models, the excellent agreement indicates that this recently developed coupled-mode model is very accurate.

It is worth noting that for the coupled-mode approaches developed by Fawcett,[10] Godin,[11] and Athanassoulis *et al.*,[12] special treatments were taken regarding the boundary condition at the bottom. In the formulations in the cited papers, the series of standard local vertical modes converges very slowly, since at a rigid bottom (or at a penetrable interface) these modes satisfy an incorrect Neumann boundary condition, wherein the normal derivative operator $\partial/\partial n$ is replaced by the derivative $\partial/\partial z$ in the vertical direction. To solve this problem, additional terms have been introduced in the series. However, this problem does not exist in a stepwise coupled-mode method, where the waveguide is range-independent in each segment. This is verified by the numerical results shown in Fig. 3, where excellent agreement between the analytical solution and the numerical stepwise coupled-mode model is observed.

The proposed exact analytical solution and the numerical model with the use of the closed-form coupling matrices are valid only for the ideal wedge problem which involves a homogeneous water column, a pressure-release surface, a rigid bottom, and a line source parallel to the wedge apex. They cannot be used to analyze certain physical phenomena such as horizontal refraction and interaction between normal modes of discrete and continuous spectrum, which are common for sound propagation in a continental shelf environment in reality. Despite such limitations, the proposed analytical solution and the numerical model developed in Refs. [15,16] can find useful applications in underwater acoustics, especially, to check and evaluate other numerical propagation models when coupling between discrete normal modes is strong but horizontal refraction and coupling to continuous spectrum is negligible.

In summary, an analytical solution to the ideal wedge problem involving a pressure-release sea surface, a rigid bottom, a homogeneous water column, and a line source parallel to the wedge apex is proposed, which is extremely useful as a primary range-dependent benchmark. Because it provides exact solutions, it may be used to establish the accuracy of numerical propagation models. In addition, a recently developed stepwise coupled-mode model[15,16] is applied to this ideal wedge problem. Closed-form expressions for coupling matrices for propagation in waveguides bounded above by a pressure-release sea surface and below by a rigid bottom are also given. The excellent agreement between the analytical solution and the numerical model establishes the accuracy of the latter. Consequently, this numerical model can serve as a secondary benchmark against other numerical models. Theoretically, as also confirmed by numerical simulations, the proposed analytical solution is valid for all frequencies and slope angles. Moreover, the numerical model with the use of the proposed closed-form expressions for the coupling matrices applies to general range-dependent waveguides (not necessarily a wedge) consisting of a homogeneous water column, a pressure-release surface, and a rigid bottom.

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