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The Darboux Transformation and New Explicit Solutions for the Belov–Chaltikian Lattice *

XUE Bo(薛波)**, WANG Xin(王鑫)

Department of Mathematics, Zhengzhou University, Zhengzhou 450001

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An N-fold Darboux transformation with multi-parameters for the famous Belov–Chaltikian lattice is derived with the aid of the gauge transformation between the corresponding discrete 3×3 matrix spectra. By using the Darboux transformation and the reduction technique, new multi-soliton solutions for the Belov–Chaltikian lattice, which are proved to be of solitary features, are given in the exponential form.

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In recent years, because of their occurrence in various scientific fields, differential-difference equations (DDEs) have aroused the increasing interest of researchers.[1–4] DDEs have effectively described numerous complicated phenomena that occur in nature, such as quantum spin chains, the vibration of a particle in a lattice, and the pulse vibration in biological chains. Searching for solutions to the DDEs has already been an important and complicated task. So far, several systematic approaches have been proposed to obtain explicit solutions, for instance, the inverse scattering transform (IST),[5] the nonlinearization approach to the eigenvalue problem,[6] the Hirota technique,[7] the Darboux transformation (DT),[8] etc. Among them, DT is one of the most powerful and fruitful approaches.

In this Letter, we search for explicit solutions for the celebrated Belov–Chaltikian (BC) lattice:

$$\begin{aligned} u_{n,t} &= u_n(u_{n+1}-u_{n-1})-(w_{n+1}-w_n),\\ w_{n,t} &= w_n(u_{n+1}-u_{n-2}), \end{aligned} \tag{1}$$

which were originally introduced in the study of lattice analogues of W-algebras by Belov and Chaltikian in 1993.[9] Some resultful methods have been applied to the BC lattice to derive its explicit solutions over recent years, such as the Bäcklund transformation,[10] and the ADM-Padě technique.[11] However, compared to 2×2 matrix spectral problems, the DT associated with 3×3 matrix spectral problems, especially the discrete type, has received less attention. This has motivated us to take advantage of this classical technique to seek for the diverse types of solutions for the BC lattice.

The aim of the present study is to construct a DT for the BC lattice with the aid of a gauge transformation between the corresponding discrete 3×3 matrix spectral problems. Then some new explicit solutions of the BC lattice are provided according to the DT derived here.

In what follows, we construct a Darboux transformation for the BC lattice, which has the Lax representation[12]

$$\psi_{n+1}=U_n\psi_n,\ U_n=\begin{pmatrix} 0 & \zeta & 0\\ -u_n & 1 & w_n\\ \zeta & 0 & 0\end{pmatrix}, \tag{2}$$

$$\psi_{n,t}=V_n\psi_n,\ V_n=\begin{pmatrix} u_{n-1} & 1 & 0\\ -u_n\zeta^{-1} & \zeta^{-1}+u_n & w_n\zeta^{-1}\\ 1 & 0 & u_{n-2}\end{pmatrix}, \tag{3}$$

where u_n and w_n are potentials, ζ a spectral parameter. Spontaneously, the zero-curvature equation $U_{n,t}=V_{n+1}U_n-U_nV_n$ yields the BC lattice by means of a direct computation.

We firstly introduce the gauge transformation for the spectral problems (2) and (3),

$$\overline{\psi}_n=T_n\psi_n, \tag{4}$$

then Eqs. (2) and (3) are transformed into a new Lax pairs,

$$\overline{\psi}_{n+1}=\overline{U}_n\overline{\psi}_n,\qquad \overline{\psi}_{n,t}=\overline{V}_n\overline{\psi}_n, \tag{5}$$

where

$$\overline{U}_n=T_{n+1}U_nT_n^{-1},\qquad \overline{V}_n=(T_{n,t}+T_nV_n)T_n^{-1}. \tag{6}$$

If the above new spectral problem is the same type as the original one, we call the gauge transformation the Darboux transformation.

Suppose

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**Corresponding author. Email: xuebo@zzu.edu.cn.

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$$T_n = \begin{pmatrix} (1-B_n^{(1)}) + \sum_{k=1}^{N} A_n^{(k)}\zeta^k & \sum_{k=1}^{N} B_n^{(k)}\zeta^k & \sum_{k=0}^{N-1} C_n^{(k)}\zeta^k \\ \sum_{k=0}^{N-1} D_n^{(k)}\zeta^k & 1+\sum_{k=1}^{N} E_n^{(k)}\zeta^k & \sum_{k=0}^{N-1} F_n^{(k)}\zeta^k \\ -H_n^{(1)} + \sum_{k=1}^{N} G_n^{(k)}\zeta^k & \sum_{k=1}^{N} H_n^{(k)}\zeta^k & A_{n-1}^{(N)}\zeta^N + \sum_{k=0}^{N-1} I_n^{(k)}\zeta^k \end{pmatrix}, \tag{7}$$

where $A_n^{(l)}, B_n^{(l)}, C_n^{(l)}, D_n^{(l)}, E_n^{(l)}, F_n^{(l)}, G_n^{(l)}, H_n^{(l)}, I_n^{(l)}$ are functions of n and t. Let $\varphi_n = (\varphi_n^1, \varphi_n^2, \varphi_n^3)^{\mathrm{T}}$, $\psi_n = (\psi_n^1, \psi_n^2, \psi_n^3)^{\mathrm{T}}$, $\chi_n = (\chi_n^1, \chi_n^2, \chi_n^3)^{\mathrm{T}}$ are three basic solutions of the spectral problems (2), we write them as $\Phi_n = (\varphi_n, \psi_n, \chi_n)^{\mathrm{T}}$, then $\overline{\Phi}_n = T_n \Phi_n$. Assume $\zeta_1, \zeta_2, \cdots, \zeta_{3N}$ $(\zeta_i \neq \zeta_k, i \neq k)$ to be $3N$ arbitrary given parameters and the roots of $\det T_n$, then as $\zeta = \zeta_j$, the column vectors of $\overline{\Phi}_n$ are linearly dependent, i.e., there exits constants $\gamma_j^{(1)}, \gamma_j^{(2)}$ $(j = 1, 2, \cdots, 3N)$, which satisfy

$$\begin{aligned}
&\sum_{k=1}^{N} A_n^{(k)}\zeta_j^k + (\zeta_j\alpha_j^{(1)}(n) - 1)B_n^{(1)} + \alpha_j^{(1)}(n)\sum_{k=2}^{N} B_n^{(k)}\zeta_j^k \\
&+ \alpha_j^{(2)}(n)\sum_{k=0}^{N-1} C_n^{(k)}\zeta_j^k = -1, \\
&\sum_{k=0}^{N-1} D_n^{(k)}\zeta_j^k + \alpha_j^{(1)}(n)\sum_{k=1}^{N} E_n^{(k)}\zeta_j^k \\
&+ \alpha_j^{(2)}(n)\sum_{k=0}^{N-1} F_n^{(k)}\zeta_j^k = -\alpha_j^{(1)}(n), \\
&\sum_{k=1}^{N} G_n^{(k)}\zeta_j^k + (\zeta_j\alpha_j^{(1)}(n) - 1)H_n^{(1)} + \alpha_j^{(1)}(n)\sum_{k=2}^{N} H_n^{(k)}\zeta_j^k \\
&+ \alpha_j^{(2)}(n)\sum_{k=0}^{N-1} I_n^{(k)}\zeta_j^k = -\alpha_j^{(2)}(n)A_{n-1}^{(N)}\zeta_j^N,
\end{aligned} \tag{8}$$

with

$$\begin{aligned}
\alpha_j^{(1)}(n) &= \frac{\varphi_n^2(\zeta_j) + \gamma_j^{(1)}\psi_n^2(\zeta_j) + \gamma_j^{(2)}\chi_n^2(\zeta_j)}{\varphi_n^1(\zeta_j) + \gamma_j^{(1)}\psi_n^1(\zeta_j) + \gamma_j^{(2)}\chi_n^1(\zeta_j)}, \\
\alpha_j^{(2)}(n) &= \frac{\varphi_n^3(\zeta_j) + \gamma_j^{(1)}\psi_n^3(\zeta_j) + \gamma_j^{(2)}\chi_n^3(\zeta_j)}{\varphi_n^1(\zeta_j) + \gamma_j^{(1)}\psi_n^1(\zeta_j) + \gamma_j^{(2)}\chi_n^1(\zeta_j)},
\end{aligned} \tag{9}$$

where $\gamma_j^{(1)}, \gamma_j^{(2)}$ and ζ_j $(j = 1, 2, \cdots, 3N)$ are suitably chosen so that any determinant of the coefficient for Eq. (8) are unequal to zero. Hence, T_n is uniquely determined by linear algebraic system (8).

Theorem 1: The matrix $\overline{U}_n$ determined by the first expression of Eq. (6) has the same form as U_n, where the transformation formula from the original potential into new ones is given by

$$\begin{aligned}
\overline{u}_n &= -\frac{F_{n+1}^{(N-1)} - E_{n+1}^{(N)}u_n}{A_n^{(N)}} + \frac{G_n^{(N)}E_{n+1}^{(N)}}{A_n^{(N)}A_{n-1}^{(N)}}w_n, \\
\overline{w}_n &= \frac{E_{n+1}^{(N)}}{A_{n-1}^{(N)}}w_n.
\end{aligned} \tag{10}$$

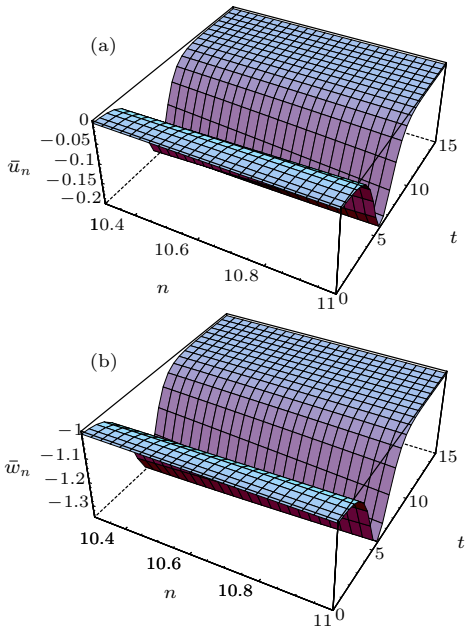


Fig. 1. One-soliton solutions of (a) $\overline{u}_n$ and (b) $\overline{w}_n$ with $\gamma_1^{(1)} = 1$, $\gamma_1^{(2)} = 0$, $\gamma_2^{(1)} = 1$, $\gamma_2^{(2)} = 0$, $\gamma_3^{(1)} = 1$, $\gamma_3^{(2)} = 0$, $\zeta_1 = 0.13$, $\zeta_2 = 0.23$, $\zeta_3 = -0.33$.

Proof: Let $T_n^{-1} = T_n^*/\det T_n$ and

$$T_{n+1}U_nT_n^* = (f_{sl}(\zeta, n))_{3\times 3}. \tag{11}$$

It is easy to see that $f_{sl}(\zeta, n)(s, l = 1, 2, 3)$ are $(3N)$th-order or $(3N+1)$th-order polynomials in ζ, Meanwhile, combining Eqs. (2) and (9), we obtain

$$\begin{aligned}
\alpha_j^{(1)}(n+1) &= \frac{-u_n + \alpha_j^{(1)}(n) + w_n\alpha_j^{(2)}(n)}{\zeta_j\alpha_j^{(1)}(n)}, \\
\alpha_j^{(2)}(n+1) &= \frac{1}{\alpha_j^{(1)}(n)}.
\end{aligned} \tag{12}$$

Through direct computation, it can be verified that all $\zeta_j (1 \leq j \leq 3N)$ are roots of $f_{sl}(\zeta, n)(s, l = 1, 2, 3)$. Then we have

$$T_{n+1}U_nT_n^* = (\det T_n)P(\zeta, n) \tag{13}$$

with

$$P(\zeta, n) = \begin{pmatrix} P_{11}^{(0)} & P_{12}^{(1)}\zeta & P_{13}^{(0)} \\ P_{21}^{(0)} & P_{22}^{(0)} & P_{23}^{(0)} \\ P_{31}^{(1)}\zeta & P_{32}^{(0)} & P_{33}^{(0)} \end{pmatrix}.$$

Comparing the coefficients of ζ^0, ζ^1, ζ^N, ζ^{N+1} in Eq. (13), we can arrive at

$$\begin{aligned}
&P_{11}^{(0)} = 0, \quad P_{12}^{(1)} = 1, \quad P_{13}^{(0)} = 0, \\
&P_{21}^{(0)} = -\overline{u}_n, \quad P_{22}^{(0)} = 1, \quad P_{23}^{(0)}(n) = \overline{w}_n, \\
&P_{31}^{(1)} = 1, \quad P_{32}^{(0)} = 0, \quad P_{33}^{(0)} = 0.
\end{aligned}$$

The proof is completed.

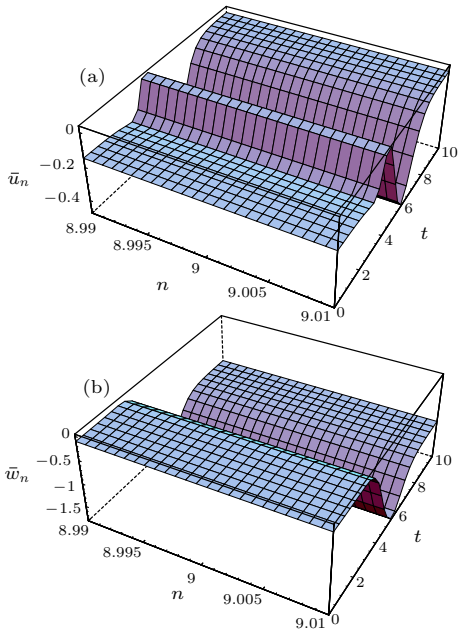


Fig. 2. One-soliton solutions of (a) $\overline{u}_n$ and (b) $\overline{w}_n$ with $\gamma_1^{(1)}=1,\ \gamma_1^{(2)}=0,\ \gamma_2^{(1)}=1,\ \gamma_2^{(2)}=0,\ \gamma_3^{(1)}=1,\ \gamma_3^{(2)}=0,\ \zeta_1=-0.18,\ \zeta_2=-0.28,\ \zeta_3=0.38.$

Theorem 2: The matrix $\overline{V}_n$ determined by the second expression of Eq. (6) has the same form as V_n, in which the old potentials are mapped into new ones according to the same DT (10).

Proof: By using Eqs. (3) and (9), we obtain

$$\begin{aligned}\alpha_{j,t}^{(1)}(n) =& -u_n\zeta_j^{-1}+(\zeta_j^{-1}+u_n-u_{n-1})\alpha_j^{(1)}(n)\\ &+w_n\zeta_j^{-1}\alpha_j^{(2)}(n)-(\alpha_j^{(1)}(n))^2,\\ \alpha_{j,t}^{(2)}(n) =& 1+(u_{n-2}-u_{n-1})\alpha_j^{(2)}(n)-\alpha_j^{(1)}(n)\alpha_j^{(2)}(n).\end{aligned} \tag{14}$$

Then theorem 2 can be proved in a similar way to theorem 1.

Theorem 3: Every solution (u_n, w_n) of the BC lattice (1) is mapped into a new solution $(\overline{u}_n, \overline{w}_n)$ of Eq. (1) under the DT (4) and Eq. (10), where T_n is uniquely given by linear algebraic system (8).

After constructing the DT for the BC lattice equation, we use the DT to generate new explicit solutions for Eq. (1). Substituting the trivial solution $u_n=0$, $w_n=-1$ into Eqs. (2) and (3), the matrix of the basic solution can be chosen as

$$\Phi(\zeta_j)=\begin{pmatrix} k_{j1}^{n+1}e^{(k_{j1}/\zeta_j)t} & k_{j2}^{n+1}e^{(k_{j2}/\zeta_j)t} & k_{j3}^{n+1}e^{(k_{j3}/\zeta_j)t}\\ \zeta_j^{-1}k_{j1}^{n+2}e^{(k_{j1}/\zeta_j)t} & \zeta_j^{-1}k_{j2}^{n+2}e^{(k_{j2}/\zeta_j)t} & \zeta_j^{-1}k_{j3}^{n+2}e^{(k_{j3}/\zeta_j)t}\\ \zeta_j k_{j1}^{n}e^{(k_{j1}/\zeta_j)t} & \zeta_j k_{j2}^{n}e^{(k_{j2}/\zeta_j)t} & \zeta_j k_{j3}^{n}e^{(k_{j3}/\zeta_j)t}\end{pmatrix}, \tag{15}$$

where k_{j1}, k_{j2}, k_{j3} $(1\le j\le 3N)$ are three different real roots of the third-order polynomial equation $k^3-k^2+\zeta_j^2=0$ $(0<|\zeta_j|<\sqrt{\frac{4}{27}})$, and $k_{j3}<0<k_{j1}<k_{j2}$. Then we have

$$\begin{aligned}\alpha_j^{(1)}(n) =& \big[\zeta_j^{-1}k_{j1}+\gamma_j^{(1)}\zeta_j^{-1}k_{j2}\varrho_{j1}^{n+1}\exp(\zeta_j^{-1}\vartheta_{j1}t)\\ &+\gamma_j^{(2)}\zeta_j^{-1}k_{j3}\varrho_{j2}^{n+1}\exp(\zeta_j^{-1}\vartheta_{j2}t)\big]\\ &\cdot\big\{1+\gamma_j^{(1)}\varrho_{j1}^{n+1}\exp(\zeta_j^{-1}\vartheta_{j1}t)\\ &+\gamma_j^{(2)}\varrho_{j2}^{n+1}\exp(\zeta_j^{-1}\vartheta_{j2}t)\big\}^{-1},\\ \alpha_j^{(2)}(n) =& \big[\zeta_j+\gamma_j^{(1)}\zeta_j\varrho_{j1}^{n}\exp(\zeta_j^{-1}\vartheta_{j1}t)\\ &+\gamma_j^{(2)}\zeta_j\varrho_{j2}^{n}\exp(\zeta_j^{-1}\vartheta_{j2}t)\big]\\ &\cdot\big\{k_{j1}+\gamma_j^{(1)}k_{j2}\varrho_{j1}^{n}\exp(\zeta_j^{-1}\vartheta_{j1}t)\\ &+\gamma_j^{(2)}k_{j3}\varrho_{j2}^{n}\exp(\zeta_j^{-1}\vartheta_{j2}t)\big\}^{-1},\end{aligned} \tag{16}$$

where $k_{j2}/k_{j1}=\varrho_{j1}$, $k_{j2}-k_{j1}=\vartheta_{j1}$, $k_{j3}/k_{j1}=\varrho_{j2}$, $k_{j3}-k_{j1}=\vartheta_{j2}$.

In what follows, we discuss the special case $N=1$ for simplicity. Solving the linear algebraic system (8), we have

$$F_{n+1}^{(0)}=\frac{\Delta_{2,3}^{n+1}}{\Delta_2^{n+1}},\ E_{n+1}^{(1)}=\frac{\Delta_{2,2}^{n+1}}{\Delta_2^{n+1}},\ A_n^{(1)}=\frac{\Delta_{1,1}^{n}}{\Delta_1^{n}},$$

$$G_n^{(1)}=\frac{\Delta_{3,1,0}^{n}\Delta_{1,1}^{n-1}}{\Delta_1^{n}\Delta_1^{n-1}},\ I_n^{(1)}=A_{n-1}^{N}=\frac{\Delta_{1,1}^{n-1}}{\Delta_1^{n-1}}$$

with

$$\Delta_1^n=\begin{bmatrix}\zeta_1 & \zeta_1\alpha_1^{(1)}-1 & \alpha_1^{(2)}\\ \zeta_2 & \zeta_2\alpha_2^{(1)}-1 & \alpha_2^{(2)}\\ \zeta_3 & \zeta_3\alpha_3^{(1)}-1 & \alpha_3^{(2)}\end{bmatrix},$$

$$\Delta_2^n=\begin{bmatrix}1 & \zeta_1\alpha_1^{(1)} & \alpha_1^{(2)}\\ 1 & \zeta_2\alpha_2^{(1)} & \alpha_2^{(2)}\\ 1 & \zeta_3\alpha_3^{(1)} & \alpha_3^{(2)}\end{bmatrix},$$

$$\Delta_{1,1}^n=\begin{bmatrix}-1 & \zeta_1\alpha_1^{(1)}-1 & \alpha_1^{(2)}\\ -1 & \zeta_2\alpha_2^{(1)}-1 & \alpha_2^{(2)}\\ -1 & \zeta_3\alpha_3^{(1)}-1 & \alpha_3^{(2)}\end{bmatrix},$$

$$\Delta_{2,2}^n=\begin{bmatrix}1 & -\alpha_1^{(1)} & \alpha_1^{(2)}\\ 1 & -\alpha_2^{(1)} & \alpha_2^{(2)}\\ 1 & -\alpha_3^{(1)} & \alpha_3^{(2)}\end{bmatrix},$$

$$\Delta_{2,3}^n=\begin{bmatrix}1 & \zeta_1\alpha_1^{(1)} & -\alpha_1^{(1)}\\ 1 & \zeta_2\alpha_2^{(1)} & -\alpha_2^{(1)}\\ 1 & \zeta_3\alpha_3^{(1)} & -\alpha_3^{(1)}\end{bmatrix},$$

$$\Delta_{3,1,0}^n=\begin{bmatrix}-\zeta_1\alpha_1^{(2)} & \zeta_1\alpha_1^{(1)}-1 & \alpha_1^{(2)}\\ -\zeta_2\alpha_2^{(2)} & \zeta_2\alpha_2^{(1)}-1 & \alpha_2^{(2)}\\ -\zeta_3\alpha_3^{(2)} & \zeta_3\alpha_3^{(1)}-1 & \alpha_3^{(2)}\end{bmatrix}.$$

Then it follows

$$\overline{u}_n = -\frac{\Delta_{2,3}^{n+1}\Delta_1^n}{\Delta_{1,1}^n\Delta_2^{n+1}} - \frac{\Delta_{3,1,0}^n\Delta_{2,2}^{n+1}}{\Delta_{1,1}^n\Delta_2^{n+1}},\ \overline{w}_n = -\frac{\Delta_{2,2}^{n+1}\Delta_1^{n-1}}{\Delta_2^{n+1}\Delta_{1,1}^{n-1}}. \tag{17}$$

In the same way, we can obtain more new explicit solutions to Eq. (1) one by one. Finally, some new soliton solutions are displayed by suitably choosing the parameters.

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