Generalized Synchronization of Diverse Structure Chaotic Systems *

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Generalized synchronization between two diverse structures of chaotic systems possesses significance in the research of synchronization. We propose an approach based on the Lyapunov stability theory to study it. This method can be used widely. Numerical examples are given to demonstrate the effectiveness of this approach.

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The potential for identical and sufficient strongly coupled chaotic systems to synchronize is well known.[1,2] This phenomenon is important theoretically because it is not obvious that systems with sensitive dependence on initial conditions should be able to synchronize. It is also of possible practical importance in fields such as communications,[3] neuroscience[4] and ecology.[5]

Three types of synchronization are frequently discussed. The strongest form, i.e. complete synchronization (CS), means that the distinct oscillators asymptotically approach the same orbit. Phase synchronization (PS) refers to the exact entrainment of the phase of distinct oscillators, but with no restriction on their amplitudes. Generalized synchronization (GS), which is normally defined only for systems involving "driving" and "receiving" components, refers to a situation in which the states of the "driver" and the "receiver" on their respective attractors are related by a continuous mapping from the driver state to the receiver state.[6] CS is, of course, the special case of GS in which the relevant mapping is the identity transformation. PS does not imply GS because the oscillator amplitudes can be uncorrelated in the case of PS. GS does not imply PS because it is not necessarily possible even to define the phase of a chaotic oscillator. Each of these types of synchronization has been shown to occur in theoretical models,[7] and in experiments[8] as a result of deterministic coupling between the oscillators. In addition, CS and PS have been shown to occur in uncoupled systems that are subject to common noise.[9]

At present, most of the theoretical results about synchronization phenomena focus on structurally equivalent systems (identical systems or nonidentical systems whose nonidentity results from a rather small parameter mismatch).[7] In this Letter, we propose an approach to realize the globally GS between diverse structures of chaotic systems.

We consider the chaotic system described by

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{f}(\boldsymbol{x}(k), k), \tag{1}$$

and the response system of Eq. (1) defined by

$$\boldsymbol{y}(k+1) = \boldsymbol{B}\boldsymbol{y}(k) + \boldsymbol{g}(y(k), k) + \boldsymbol{u}(y(k), y(k), k), \tag{2}$$

where $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, $y = (y_1, y_2, \ldots, y_m)^{\mathrm{T}}$, $\boldsymbol{A} \in \boldsymbol{R}^{n\times n}$ and $\boldsymbol{B} \in \boldsymbol{R}^{m\times m}$ are constant matrixes, $\boldsymbol{f}(\boldsymbol{x}(k), k) : R^n \times \boldsymbol{N} \to \boldsymbol{R}^n$ and $\boldsymbol{g}(\boldsymbol{y}(k), k) : \boldsymbol{R}^m \times \boldsymbol{N} \to \boldsymbol{R}^m$ are nonlinear (or piecewise-linear) vector functions, $u(x(k), y(k), k) \in R^m$ is the vector controller, and k is the iteration index.

Definition 1: We say systems (1) and (2) are GS with respect to the vector map φ, if there exist a controller $u\,(x\,(k)\,, y\,(k)\,, k) \in R^m$ and a given map $\varphi : R^n \to R^m$ such that the solutions of systems (1) and (2) satisfy the property:

$$\lim_{k\to\infty} ||y(k) - \varphi(x(k))|| = 0, \tag{3}$$

where $||\cdot||$ is the Euclidean norm of a vector.

Remark 2: There are many exact classes of GS based on the mathematical property of the map φ.[10] Here the map φ is an arbitrary continuously differentiable function. Thus, with this definition, our study on GS has more extensive meaning than before.

Defining the term $e(k) = y(k) - \varphi(x(k))$ as an error between systems (1) and (2) at iteration k, thus the error dynamics between the drive system (1) and

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the response system (2) can be derived as

$$\begin{aligned} e(k+1) =&\Delta e(k) = \Delta y(k) - \Delta\varphi(x(k)) \\ =&y(k+1) - D\varphi(x(k))x(k+1) \\ =&By(k) + g(y(k),k) + u(x(k),y(k),k) \\ &- D\varphi(x(k))x(k+1), \end{aligned} \tag{4}$$

where $D\varphi(x(k))$ is the Jacobian matrix of the map $\varphi(x(k))$.

$$D\varphi(x(k)) = \begin{bmatrix} \frac{\partial\varphi_1(x(k))}{\partial x_1} & \frac{\partial\varphi_1(x(k))}{\partial x_2} & \cdots & \frac{\partial\varphi_1(x(k))}{\partial x_n} \\ \frac{\partial\varphi_2(x(k))}{\partial x_1} & \frac{\partial\varphi_2(x(k))}{\partial x_2} & \cdots & \frac{\partial\varphi_2(x(k))}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial\varphi_m(x(k))}{\partial x_1} & \frac{\partial\varphi_m(x(k))}{\partial x_2} & \cdots & \frac{\partial\varphi_m(x(k))}{\partial x_n} \end{bmatrix}. \tag{5}$$

From the definition of GS, we know that the study of the GS between systems (1) and (2) can be translated into the analysis of the asymptotical stability analysis of zero solution of error dynamics system. Therefore, we discuss the asymptotical stability of zero solution of system (4) and draw the following conclusion.

Theorem 1 In systems (1) and (2), for any continuously differentiable map $\varphi : R^n \to R^m$, let us design the controller

$$\begin{aligned} u =&K(y(k) - \varphi(x(k))) - g(y(k),k) \\ &+ D\varphi(x(k))x(k+1) - B\varphi(x(k)), \end{aligned} \tag{6}$$

where $K \in R^{m\times m}$ is the error feedback gain matrix. If the eigenvalues of matrix $K + B$ are in the unit disc except for the origin, then systems (1) and (2) are globally GS with respect to the vector map φ.

Proof: Substituting Eq. (6) into Eq. (4), we obtain

$$e(k+1) = Ke(k) + Be(k) = Qe(k), \tag{7}$$

where

$$Q = K+B = \begin{bmatrix} k_{11}+b_{11} & k_{12}+b_{12}\cdots k_{1m}+b_{1m} \\ k_{21}+b_{21} & k_{22}+b_{22}\cdots k_{2m}+b_{2m} \\ \vdots & \vdots & \vdots \\ k_{m1}+b_{m1} & k_{m2}+b_{m2}\cdots k_{mm}+b_{mm} \end{bmatrix}. \tag{8}$$

In view of the form of Q, we can find that for any given parameters b_{ij} in Eq. (2), there always exists at least one family of parameters k_{ij} such that all eigenvalues of matrix Q are in the unit disc except for the origin, which guarantees the global asymptotical stability of the error system (7). For convenience, we let case 1 read

$$\begin{aligned} &0 < |k_{ij} + b_{ij}| < 1, \quad i \le j, \\ &k_{ij} = -b_{ij}, \qquad\quad i > j, \end{aligned} \tag{9}$$

or case 2 read

$$\begin{aligned} &0 < |k_{ij} + b_{ij}| < 1, \quad i \ge j, \\ &k_{ij} = -b_{ij}, \qquad\quad i < j. \end{aligned} \tag{10}$$

In addition, we can choose the Lyapunov function as follows: $V(k) = \sum_{i=1}^{m} |e_i(k)|$, then $V(k) \ge 0$ obviously. The derivative of $V(k)$ reads

$$\begin{aligned} \Delta V(k) =&V(k+1) - V(k) = \sum_{i=1}^{m} \Big|\sum_{j=1}^{m} (k_{ij}+b_{ij})e_j(k)\Big| \\ &- \sum_{i=1}^{m} |e_i(k)| \le \sum_{i=1}^{m} (\sum_{j=1}^{m} |k_{ij} + b_{ij}| - 1)|e_i(k)|. \end{aligned} \tag{11}$$

Therefore, let

$$\sum_{j=1}^{m} |k_{ij} + b_{ij}| < 1, \ (i = 1, 2, \ldots, m) \tag{12}$$

then $\Delta V\,(k) < 0$ except for the origin. Equation (12) implies that all eigenvalues of matrix Q are in the unit disc except for the origin. According to the above discussion, in all these three cases, based on Lyapunov stability theory, the zero solution of error system (7) is globally asymptotically stable, i.e., systems (1) and (2) are globally GS with respect to the vector map φ.

For a given vector map φ, the linear controller usually cannot be found to achieve GS between two diverse structure chaotic systems. The controller should contain the information of f, g and φ. From the view of mechanical control, our controller seems to be slightly complicated, but it is appropriate for biological and social systems, and we also note that a similar controller has been adopted in Ref. [11]. The following examples will show that our approach is effective.

To demonstrate the effectiveness of our approach, we discuss two typical examples.

Example 1: We choose a 3D discrete-time Grassi–Miller map[12] as a drive system and a 2D Hénon map as the response system. The 3D discrete-time Grassi–Miller map can be described as

$$\begin{aligned} x_1(k+1) &= -0.1x_3(k) - x_2^2(k) + 1.76, \\ x_2(k+1) &= x_1(k), \\ x_3(k+1) &= x_2(k). \end{aligned} \tag{13}$$

The 2D Hénon map is given by

$$\begin{aligned} y_1(k+1) &= 1 - 1.4y_1^2(k) + y_2(k), \\ y_2(k+1) &= 0.3y_1(k). \end{aligned} \tag{14}$$

System (14) can be rewritten as a form of system (2), where

$$B = \begin{bmatrix} 0 & 1 \\ 0.3 & 0 \end{bmatrix}, \quad g(y(k),k) = \begin{bmatrix} 1 - 1.4y_1^2(k) \\ 0 \end{bmatrix}.$$

Let

$$\phi(x(k)) = [2x_1(k), \quad -0.5x_2(k) + x_3(k)]^{\mathrm{T}}, \qquad (15)$$

thus

$$D\phi(x(k)) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -0.5 & 1 \end{bmatrix}.$$

According to the above mentioned scheme,

$$Q = K + B = \begin{bmatrix} k_{11} & 1 + k_{12} \\ 0.3 + k_{21} & k_{22} \end{bmatrix},$$

and the feedback gain matrix is chosen as

$$K = \begin{bmatrix} 0.5 & -0.4 \\ -0.3 & 0.6 \end{bmatrix}.$$

It is easy to see that all eigenvalues of matrix Q are in the unit disc. Then the conditions of theorem 1 are satisfied.

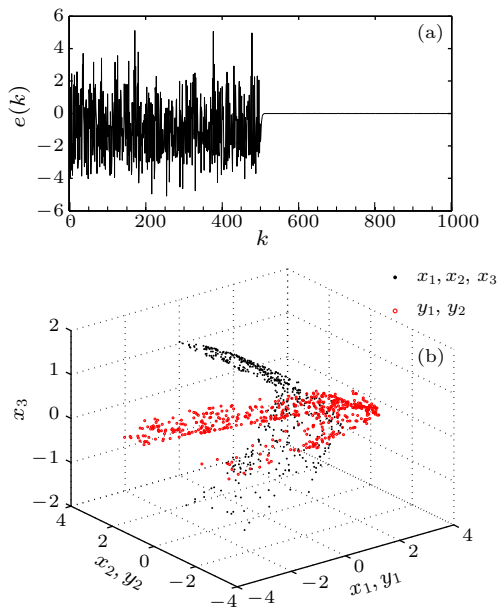

Fig. 1. The simulation results of systems (13) and (14): (a) the generalized synchronization error $e(k)$, (b) the attractors of systems (13) and (14) (red) with the map φ.

To explicitly show the control effect of the proposed method, the time interval is divided into two parts. In the first stage of the simulation, no control input is applied; the control input is then activated thereafter. With the help of Matlab, we obtain the numeric results shown in Fig. 1, where $(x_1(0), x_2(0), x_3(0)) = (0.1, 0.72, 0.8)$, $(y_1(0), y_2(0)) = (1.2, -0.2)$, and $e(k) = e_1(k) + e_2(k)$. The control input is activated after 500 iterations.

Example 2: We choose the 3D Lü chaotic system[13] as the drive system and the 4D Hyperchaotic Rössler system[14] as the response system.

The Lü chaotic system is described as

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1), \\ \dot{x}_2 &= -x_1x_3 + cx_2, \\ \dot{x}_3 &= x_1x_2 - bx_3, \end{aligned} \qquad (16)$$

where a, b, c are parameters. When $\alpha = 36$, $b = 3$, $c = 20$, system (16) is chaotic.

Rewriting it in the form of Eq. (1), we can obtain

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1x_3 \\ x_1x_2 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & 0 & 0 \\ 0 & 0 & x_2 \\ 0 & -x_3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

By using system expansion, the following form is obtained,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & 0 & 0 \\ 0 & 0 & x_2 \\ 0 & -x_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \qquad (17)$$

The hyperchaotic Rössler system[24] is described as

$$\begin{aligned} \dot{y}_1 &= \beta y_2 + \gamma y_1 + u_1, \\ \dot{y}_2 &= 3 + y_2y_3 + u_2, \\ \dot{y}_3 &= -y_2 - y_4 + u_3, \\ \dot{y}_4 &= y_1 + y_3 + \delta y_4 + u_4, \end{aligned} \qquad (18)$$

where δ, β and γ are parameters, and u_1, u_2, u_3 and u_4 are controllers added to the corresponding state vectors. The matrixes K and B are chosen as $K = diag(1, 3, 4, 2)$, $B = diag(7, 5, 8)$.

The controller is designed as

$$\begin{aligned} u_1 &= -\beta y_2 - \gamma y_1y_2 + \alpha_1(x_2 - x_1)\hat{a} - e_1, \\ u_2 &= -3 - y_2y_3 - \alpha_2x_1x_3 + \alpha_2x_2\hat{c} - 3e_2, \\ u_3 &= y_2 + y_4 + \alpha_3x_1x_2 - \alpha_3x_3\hat{b} - 4e_3, \\ u_4 &= -(y_1 + y_3 + \delta y_4) - 2e_4, \end{aligned} \qquad (19)$$

and the parameter update laws are chosen as

$$\begin{aligned} \dot{\hat{a}} &= -\alpha_1(x_2 - x_1)e_1 - 7(\hat{a} - a), \\ \dot{\hat{b}} &= \alpha_3x_3e_3 - 5(\hat{b} - b), \\ \dot{\hat{c}} &= -\alpha_2x_2e_2 - 8(\hat{c} - c). \end{aligned} \qquad (20)$$

The parameters of the response system (18) are chosen as $\delta = 0.25$, $\beta = -0.5$, $\gamma = 0.05$. The initial states of drive system (16) and response system (18) are $(x_1(0), x_2(0), x_3(0), x_4(0)) = (-1, 2, 2, 4)$, $(y_1(0), y_2(0), y_3(0), y_4(0)) = (4, 10, -3, -1)$. The initial values of unknown parameters are $\alpha(0) = 1$, $b(0) = 0$, $c(0) = 0$, and scaling factors are $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = -2$, $\alpha_4 = 1.5$, the error dynamical system reads $\dot{e}_i = \dot{y}_i - \alpha_i\dot{x}_i$ $(i = 1, 2, 3, 4)$, $e(t) = e_1(t) + e_2(t) + e_3(t) + e_4(t)$.

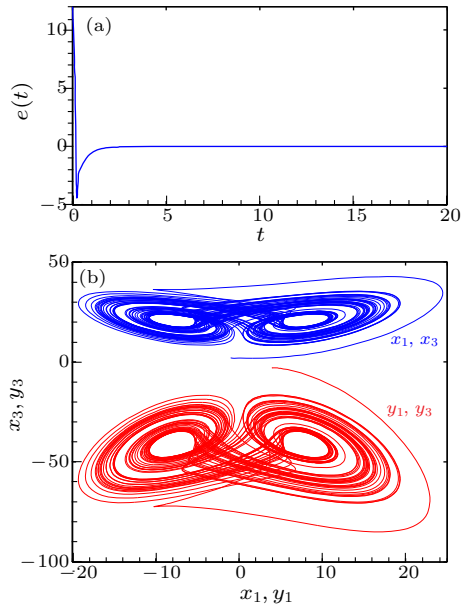

Fig. 2. The simulation results of systems (16) and (18): (a) the generalized synchronization error $e(t)$, (b) the attractors of systems (16) and (18) with the map.

With scaling factors $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = -2$, $\alpha_4 = 1.5$, the errors between drive system (16) and response system (18) and the projections of the attractors of system (16) and system (18) in R^2 are shown in Fig. 2.

As can be seen by the above example, for different structures to achieve GS of chaotic systems, the first step is to unify the dimension by reducing or increasing the dimension of the drive system and the response system, and then design the controller with the same structures. In this study, GS of diverse structure chaotic systems is investigated. Based on the Lyapunov stability theory, an effective scheme to realize GS for two diverse structure chaotic systems is proposed. By theoretical analysis, two diverse structures of chaotic systems can achieve GS by choosing reasonable feedback gain matrixes. The controller designed is robust, which can be used to achieve not only the GS of chaotic systems with the same dimension, but also the GS of chaotic systems with different dimensions. Numerical simulations are given to show the effectiveness of the proposed approach.

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