CHIN. PHYS. LETT. Vol. 28, No. 9 (2011) 090305

Experimental Realization of Braunstein's Weight-Decision Algorithm *

HOU Shi-Yao(侯世尧), CUI Jing-Xin(崔靖鑫), LI Jun-Lin(李俊林)**

State Key Laboratory of Low-dimensional Quantum Physics and Department of Physics, Tsinghua University, Beijing 100084

(Received 5 July 2011 and accepted by LONG Gui-Lu)

Braunstein proposed an algorithm to distinguish the Boolean functions of two different weights. Here we implement the algorithm in a two-qubit nuclear magnetic resonance quantum information processor. The experiment shows that the algorithm could distinguish the Boolean functions of two different weights efficiently.

PACS: 03.67.Ac, 03.67.Lx, 32.30.Dx DOI:10.1088/0256-307X/28/9/090305

A quantum algorithm can process data faster than classical ones. For example, in 1985 Deutsch proposed an algorithm to determine whether a Boolean function is balanced on one qubit,[1] and together with Jozsa, generalized the algorithm to n qubits. It offers an exponential increase in processing speed over classical algorithms.[2] Shor's algorithm[3] can factor a number exponentially faster than any classical algorithm. Grover's database search algorithm[4,5] can search a database quadratically faster than classical search algorithms.

The weight analysis of Boolean functions or a string of bits plays an important role in several fields of information processing and communication, for example, in cryptanalysis,[6] coding theory,[7] fault-tolerant circuit design[8] and built-in self-testing of circuits.[9] To efficiently distinguish the different weights of Boolean functions, Braunstein[10] analyzed the weight of a Boolean function by using a modified Grover operator that exploited phase matching in a quantum search algorithm that was established in Refs. [11–14].

In this Letter, we present the realization of an algorithm in a two-qubit NMR quantum information processor, and it shows that the algorithm can distinguish the weight of Boolean functions efficiently.

The weight of a Boolean function is defined as the ratio of the number of solutions over all possible inputs of a Boolean function. Here the weight w is one of the set of two weights $\{w_1, w_2 | w_1 + w_2 = 1, 0 < w_1 < w_2 < 1\}$. The weight decision problem is to determine the weight of a Boolean function of either w_1 or w_2, where w_1 and w_2 are already known and $w_1 + w_2 = 1$.

In Braunstein's general weight-decision algorithm, the phase of the Grover rotations should be modified. In order to distinguish the two Boolean functions, the phase of the Grover iterations should be carefully selected. By changing the phase of the Grover iterations the algorithm will distinguish the two functions deterministically. For instance, if $k = 3$ and the two weights are $w_1 = \frac{3}{8}$ and $w_2 = \frac{5}{8}$, respectively, and

$$\sin^2\left(\frac{3-1}{2\times 3-1}\frac{\pi}{2}\right) < \omega_1 \le \sin^2\left(\frac{3}{2\times 3+1}\frac{\pi}{2}\right), \quad (1)$$

three Grover iterations are needed. However, the phases in the last two iterations should be modified. The two phases of the last but one Grover iteration are θ_1 and π, and the two phases of the last Grover iteration are θ_2 and π. Here θ_1 and θ_2 satisfy

$$\cos\theta_1 = \frac{(-1)^k \cos\beta_{\omega_1} - \cos(2\beta_{\omega_1})\cos[(2k-2)\beta_{\omega_1}]}{\sin(2\beta_{\omega 1})\sin[(2k-2)\beta_{\omega_1}]}, \quad (2)$$

$$\cos\theta_2 = \frac{(-1)^k \sin(2\beta_{\omega_1})(y\sin\theta_2 - (-1)^k \sin\beta_{\omega_1})}{\cos\beta_{\omega_1}\cos(2\beta_{\omega 1}) - (-1)^k \cos[(2k-2)\beta_{\omega_1}]}. \quad (3)$$

After three Grover iterations, one can distinguish the weight of the different Boolean functions, that is, if $\omega = \omega_1 = \frac{3}{8}$, we will get the solution state, while if $\omega = \omega_2 = \frac{5}{8}$, we will get the normalized non-solution state. Simply by measuring the state on the computational state we will know what the result is.

When the weights are certain special values, the weight-decision algorithm can be simplified to the limited weight-decision algorithm.

In Braunstein's limited weight-decision algorithm, by using k iterations of the Grover operation, we can decide exactly w from $\{w_1 = \sin^2(\frac{k}{2k+1}\frac{\pi}{2}), w_2 = \cos^2(\frac{k}{2k+2}\frac{\pi}{2})\}$.

As an example, if we choose $k = 1$, we can obtain $w_1 = \frac{1}{4}$ and $w_2 = \frac{3}{4}$. That is, the initial state can be written as $|\psi\rangle = \frac{1}{2}|s\rangle + \frac{\sqrt{3}}{2}|ns\rangle$, where $|s\rangle$ is the normalized solution, and $|ns\rangle$ is the non-solution basis state. In order to prepare the initial state, we need at least two qubits. Assuming that $|00\rangle$ is the solution, while the other three states $|01\rangle$, $|10\rangle$ and $|11\rangle$

*Supported by the National Natural Science Foundation of China under Grant No 10874098, the National Basic Research Program of China (2009CB929402, 2011CB9216002) and the Specialized Research Fund for the Doctoral Program of the Ministry of Education of China (20060003048).

**To whom any correspondence should be addressed. Email: center@mail.tsinghua.edu.cn

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are the non-solution basis states, this gives $|s\rangle = |00\rangle$ and $|ns\rangle = \frac{1}{\sqrt{3}}(|01\rangle + |10\rangle + |11\rangle)$.

To decide the weight of the function, we need to implement k iterations of the Grover operator. Here $k = 1$, so we need one iteration. After one iteration, the state becomes $|\psi_{w,1}\rangle = |s\rangle$. By measuring the state in the computational basis we will obtain the result $|s\rangle$.

On the other hand, if the weight w is $\frac{3}{4}$, that is, $w_1 = \frac{3}{4}$ and $w_2 = \frac{1}{4}$ and the initial state is $|\psi\rangle = \frac{\sqrt{3}}{2}|s\rangle + \frac{1}{2}|ns\rangle$. Using two qubits we can prepare the initial state by assuming $|s\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$ and $|ns\rangle = |11\rangle$. By applying the standard Grover operator once, the initial state turns into $|\psi_{w,1}\rangle = -|ns\rangle$. By measuring the state in the computational basis we will obtain the result $|ns\rangle$.

Thus simply by applying the Grover operator, we can tell whether the state is $|\psi\rangle = \frac{1}{2}|s\rangle + \frac{\sqrt{3}}{2}|ns\rangle$ or $|\psi\rangle = \frac{\sqrt{3}}{2}|s\rangle + \frac{1}{2}|ns\rangle$. That is, we know whether the weight of the function is $\frac{1}{4}$ or $\frac{3}{4}$.

Experimentally we implemented the algorithm in a two-qubit NMR quantum information processor. The NMR technique is one of the most important realization schemes of quantum computers. Various quantum algorithms and quantum communication protocols were shown on nuclear magnetic resonance systems, such as the Grover algorithm,[15] the Shor algorithm,[16] the Deustch–Jozsa algorithm,[17,18] iterative phase estimation algorithms,[19] the fixed point duality search algorithm,[20] the quantum secret sharing protocol[21] and the quantum private query algorithm.[22]

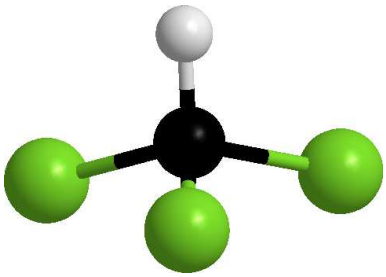

Fig. 1. The molecular structure of chloroform. The black ball represents a carbon nucleus, the gray ball represents a hydrogen nucleus, while the three green balls are chlorine nuclei, which will not affect our experiment.

In our experiment, the ^{13}C and ^{1}H in a sample of chloroform (the molecular structure of which is shown in Fig. 1) are used as the two qubits. The experiments were conducted on a Bruker Avance Ⅲ 400 MHz spectrometer at room temperature. The internal Hamiltonian of the system is

$$H = -\omega_1 I_z^1 - \omega_2 I_z^2 + 2\pi J I_z^1 I_z^2, \tag{4}$$

where ω_1 and ω_2 are the resonance frequencies of the spin of the ^{1}H nuclei and the spin of the ^{13}C nuclei, respectively, and J is the scalar coupling between the two spins. I_z^is are the spin angular momentum operators, while $I_z = \frac{1}{2}\sigma_z$, where $\sigma_i (i = x, y, z)$ represents the Pauli matrices.

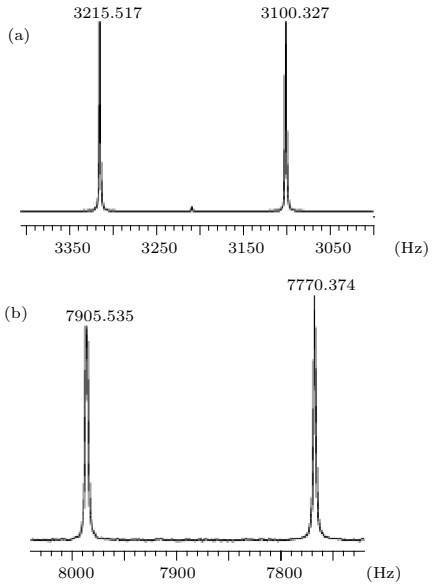

Fig. 2. The NMR spectra of the chloroform molecule, from which we can see the chemical shift of the ^{1}H(left) nuclei and ^{13}C(right)nuclei and their scalar coupling in the chloroform molecule. The chemical shift of ^{1}H is 3210 Hz, while the chemical shift of the ^{13}C is 7825 Hz. The scalar coupling J between the ^{1}H and ^{13}C is 215.19 Hz.

Before any operation, the system is in a thermal equilibrium state

$$\rho_{eq}^{*} = \gamma_1 I_z^1 + \gamma_2 I_z^2. \tag{5}$$

The spectrum of the two nuclei in the thermal equilibrium state is shown in Fig. 2.

First the pseudo-pure state is prepared, here we adopt the spatial averaging method, which was introduced in Ref. [23] with the following pulse sequence

$$[\alpha]_x^2 \to [\text{grad}]_z \to [\frac{\pi}{4}]_{-x}^2 \to [\frac{1}{2J}] \to [\frac{\pi}{4}]_y^2 \to [\text{grad}]_z, \tag{6}$$

which gives the pseudo-pure state

$$\rho = 2\gamma_1 \Big(|00\rangle\langle 00| - \frac{I}{4}\Big). \tag{7}$$

After this, applying a Hadamard gate to each of the two qubits will transfer the state to $\frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, which is the equally superposition state $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. The Hadamard operation can be realized by the following pulse sequence

$$[\frac{\pi}{2}]_y \to [\pi]_{-x}. \tag{8}$$

To distinguish the two weights, we performed the Grover iteration on the equally superposition state. The Grover operator consists of two inversions: one about the equally superposition state and the other

about the solution state. Such operations include two unitary operations and two Hadamard operations on both qubits. These operations are accomplished by the following pulses,

$$[\frac{1}{2J}] \rightarrow [\pi]^{1,2}_{-y} \rightarrow [\frac{1}{2J}] \rightarrow [\frac{\pi}{2}]^{1,2}_{x} \rightarrow [\frac{\pi}{2}]^{1,2}_{y} \rightarrow [\frac{\pi}{2}]^{1,2}_{x}. \quad (9)$$

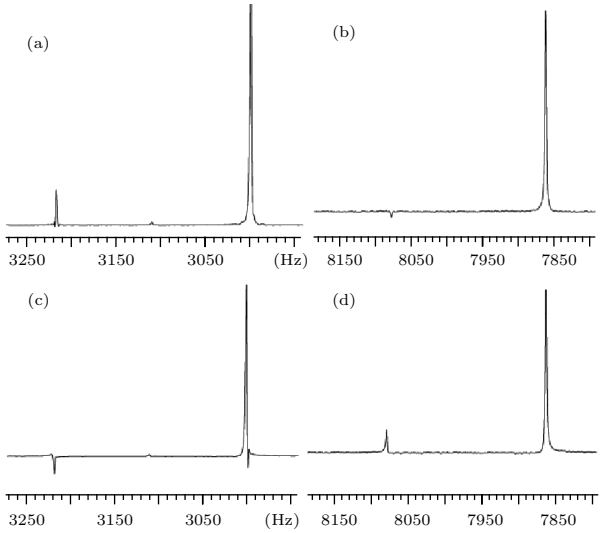

Fig. 3. The NMR spectra of the chloroform molecule. Upper left is the H spectrum and upper right is the C spectrum. Both are in the pseudo-pure state $|00\rangle$. Lower left is the H spectrum and lower right is the C spectrum. From these spectra we can easily find that the system is in the state $|s\rangle$, and the weight of the Boolean function is $\frac{1}{4}$.

From the spectra we obtain the final result. There are only two reasonable results. One is that the weight is $\frac{1}{4}$ and the final result is $|s\rangle = |00\rangle$, the other is that the weight is $\frac{3}{4}$ and the final result is $|ns\rangle = |11\rangle$, so finally from the different spectra we can distinguish the weight of the function. The result can be seen in Fig. 3.

We have completed the experimental realization of Braustein's weight-decision algorithm which is much more powerful than a classical algorithm. In our experiment, only one Grover iteration is needed while we have to examine each case four times using the classical Boolean function. We have shown that the algorithm can efficiently distinguish the weights of Boolean functions.

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