CHIN. PHYS. LETT. Vol. 28, No. 6 (2011) 068903

Local Natural Connectivity in Complex Networks

SHANG Yi-Lun(尚轶伦)**

Institute for Cyber Security, University of Texas at San Antonio, San Antonio, TX 78249, USA

(Received 24 October 2010)

In network theory, a complex network represents a system whose evolving structure and dynamic behavior contribute to its robustness. The natural connectivity is recently proposed as a spectral measure to characterize the robustness of complex networks. We decompose the natural connectivity of a network as local natural connectivity of its connected components and quantify their contributions to the network robustness. In addition, we compare the natural connectivity of a network with that of an induced subgraph of it based on interlacing theorems. As an application, we derive an inequality for eigenvalues of Erdös-Rényi random graphs.

PACS: 89.75.Hc, 89.75.Fb, 02.10.Ox DOI: 10.1088/0256-307X/28/6/068903

The study of complex networks, albeit young, has spanned diverse domains including science, biology, engineering, business, sociology and psychology, to name a few.[1–4] Complex networks rely for their function and performance on their robustness, which defines a network's survivability in the advent of typical component failures and at the onset of cryptic malicious attacks.[5–9] In network theory, robustness, as one of the most central research topics, can encompass properties ranging from redundancy and diversity, to concepts such as the ability to operate under perturbation or the efficiency of feedback control mechanisms.[10–12]

The classical approach for determining robustness of networks entails the use of basic concepts from graph theory. For instance, the connectivity of a graph is a fundamental measure of robustness of a network.[13] Node (edge) connectivity, defined as the size of the smallest node (edge) cut, describes the robustness of a graph to the deletion of nodes (edges). However, the node or edge connectivity only partly reflects the ability of graphs to retain connectedness after deletion. Other improved measures include restricted connectivity,[14] fault diameter,[15] toughness,[16] scattering number,[17] isoperimetric number,[18] and so forth. In contrast to node (edge) connectivity, these new measures treat both the cost to damage a network and how badly the network is damaged. From the perspective of percolation and random graphs,[19] the work[5] deals with statistical measures pertaining to the size of the largest connected component as nodes are attacked. In spectral analysis, the second smallest Laplacian eigenvalue (i.e. the algebraic connectivity) is addressed as a measure of how difficult it is to break a network into independent components.[20] However, for all disconnected networks the algebraic connectivity is equal to zero.

Recently, the concept of natural connectivity was proposed in Ref. [21] as a spectral measure of robustness in complex networks. The natural connectivity was expressed in mathematical form as a special case of average eigenvalue and it characterizes the redundancy of alternative paths by quantifying the weighted number of closed walks of all lengths. It was shown Ref. [21] that the natural connectivity has acute discrimination in measuring the robustness of complex networks and can exhibit the variation of robustness sensitively even for disconnected networks. The same authors further investigated the natural connectivity in the context of regular graph, random graph and Internet AS-level topology, respectively, in Refs. [22–24]. It is noteworthy that, based on the spectral techniques, the recent work[25] has examined the perturbation of natural connectivity in weighted networks.

Many phenomena in complex networks are often difficult to understand because the causes and effects are not obviously related. Pushing on a complex system "here" often has effects "over there" because the parts are interdependent. To move a step further in understanding local structure of complex networks regarding robustness issues, we focus on the natural connectivity of subgraphs which constitute the whole network, and try to characterize the correlations between the (holistic) natural connectivity and the local natural connectivity. In this Letter, we decompose the natural connectivity of a network as local natural connectivity of its connected components and quantify their contributions to the robustness of the system. Moreover, we compare the natural connectivity of a network with that of an (arbitrary) induced subgraph (not necessarily connected) of it based on interlacing theorems[26] in algebraic graph theory. A new inequality for eigenvalues of classical random graph is given as an application of our result.

A complex network can be described by a sim-

**Email: shylmath@hotmail.com

© 2011 Chinese Physical Society and IOP Publishing Ltd

ple undirected graph $G = (V, E)$, where V is the set of vertices, and $E \subseteq V \times V$ is the set of edges. Denote by $N = |V|$ the number of vertices. Let $A(G) = (a_{ij})_{N\times N}$ be the adjacency matrix of G, where $a_{ij} = a_{ji} = 1$ if vertices v_i and v_j are adjacent, and $a_{ij} = a_{ji} = 0$ otherwise. It is obvious that $A(G)$ is a real symmetric matrix and we let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ denote the eigenvalues of A which are usually also called the eigenvalues of the graph G itself. The set $\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$ is called the spectrum of G. A walk of length k in a graph G is an alternating sequence of vertices and edges $v_0e_1v_1e_2\cdots e_kv_k$, where $v_i \in V$ and $e_i = (v_{i-1}, v_i) \in E$. A walk is closed if $v_0 = v_k$.

A weighted sum of numbers of closed walks was defined in Ref. [21] by $S = \sum_{k=0}^{\infty} n_k/k!$, where n_k is the number of closed walks of length k. This scaling ensures that the weighted sum does not diverge and it also means that S can be obtained from the powers of the adjacency matrix:

$$n_k = \sum_{i=1}^{N} \lambda_i^k = \mathrm{Trace}(A^k). \tag{1}$$

Therefore, using Eq. (1) we have

$$S = \sum_{k=0}^{\infty} \frac{n_k}{k!} = \sum_{k=0}^{\infty}\sum_{i=1}^{N} \frac{\lambda_i^k}{k!} = \sum_{i=1}^{N}\sum_{k=0}^{\infty} \frac{\lambda_i^k}{k!} = \sum_{i=1}^{N} e^{\lambda_i}. \tag{2}$$

Note that Eq. (2) corresponds to the Estrada index of the graph,[27] which has been used in the investigation of bipartivity[28] and subgraph centrality.[29] The natural connectivity of G is then defined as[21]

$$\bar{\lambda} = \ln\left(\frac{1}{N}\sum_{i=1}^{N} e^{\lambda_i}\right), \tag{3}$$

which corresponds to an "average" eigenvalue of G since $\lambda_N \leq \bar{\lambda} \leq \lambda_1$.

Now suppose that the network G contains K connected components $G_1, G_2, \cdots, G_K$ having $N_1, N_2, \cdots, N_K$ vertices, respectively. Hence, $\sum_{i=1}^{K} N_i = N$, the number of vertices in the whole network. For $1 \leq i \leq K$, we denote the spectrum of G_i by $\{\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{iN_i}\}$. Therefore, the spectra of all $\{G_i\}_{i=1}^{K}$ partition the spectrum of G. Let $\bar{\lambda_i}$ be the local natural connectivity of G_i. From Eq. (3), it is shown to be given by $\bar{\lambda_i} = \ln\left(\sum_{j=1}^{N_i} e^{\lambda_{ij}}/N_i\right)$. Thus we may decompose the natural connectivity $\bar{\lambda}$ of G as follows:

$$\bar{\lambda} = \ln\left(\frac{1}{N}\sum_{j=1}^{K} N_j e^{\bar{\lambda_j}}\right). \tag{4}$$

It is evident from Eq. (4) that

$$\min\{\bar{\lambda_1}, \cdots, \bar{\lambda_K}\} \leq \bar{\lambda} \leq \max\{\bar{\lambda_1}, \cdots, \bar{\lambda_K}\}, \tag{5}$$

where every local natural connectivity $\bar{\lambda_i}$ lies between the maximal and minimal eigenvalues in the spectrum of G_i.

To unfold the influence of local natural connectivity to the robustness of the whole network G, we may rewrite the expression (4) as

$$e^{\bar{\lambda}} = \sum_{j=1}^{K} \left(\frac{N_j}{N}\right) e^{\bar{\lambda_j}}. \tag{6}$$

We observe that the natural exponential of natural connectivity $\bar{\lambda}$ can be expressed as a weighted sum of natural exponential of local natural connectivities of components. Larger components may have greater effect on the robustness of the network in view of the weighted coefficients in Eq. (6). To make things more concrete, we instantiate Eq. (6) in the special case of an Erdös-Rényi random graph $G_{N,p}$ [32] with $N = 1000$ and $p = 0.005$. The graph is divided into $K = 4$ components with sizes $N_1 = 902$, $N_2 = 47$, $N_3 = 31$ and $N_4 = 20$, respectively. Calculation of the (local) natural connectivities gives $\bar{\lambda} = 3.8$, $\bar{\lambda_1} = 3.7$, $\bar{\lambda_2} = 4.3$, $\bar{\lambda_3} = 4.8$ and $\bar{\lambda_4} = 3.5$. We can directly verify that Eqs. (5) and (6) hold.

To further illustrate the influence on robustness of local natural connectivities, we consider a scenario of edge elimination. We generate an initial network G with two components G_1 and G_2, each of which is generated with a power-law degree distribution using the BA model.[30] We let $N_1 = 3000$, $N_2 = 1000$ and average degree $\langle k\rangle = 8$.

Six edge elimination strategies are considered. These strategies falls in two categories: (i) random strategy, i.e. deleting the edges randomly; (ii) rich-rich strategy, i.e. deleting the edges connecting high-degree to high-degree vertices in descending order of d_id_j, where d_i and d_j are the degrees of the end vertices of an edge. In (i) we consider three cases: deleting edges only from component G_1; deleting edges only from component G_2; deleting edges from both G_1 and G_2 alternately. Similarly, we consider three cases in (ii). In all the above deletion processes we will keep the graphs G_1 and G_2 connected. This is easy to contrive. If $G_1(G_2)$ becomes disconnected during the procedure, we simply throw it away and start a new procedure.

In Figs. 1 and 2 we show the results for the natural connectivity $\bar{\lambda}$ corresponding to random strategy and rich-rich strategy, respectively. From Fig. 1 we can see a linear decrease of the natural connectivity, which is consistent with those observed in Refs. [21,24]. Deleting edges only in G_1 results in a fastest decrease in robustness, while deleting edges only in G_2 has a mildest effect. This agrees with our conclusion from Eq. (6). Similarly, for the rich-rich strategy employed in Fig. 2, the curve for alternative deletion strategy

lies between the two extremes. In addition, compared with that in Fig. 1, the natural connectivity $\bar{\lambda}$ drops much rapidly with the edge elimination in Fig. 2. It is believed that the edges between high-degree vertices are important.[4,31] Thus the robustness measure based on natural connectivity has strong discrimination and agrees with our intuition.

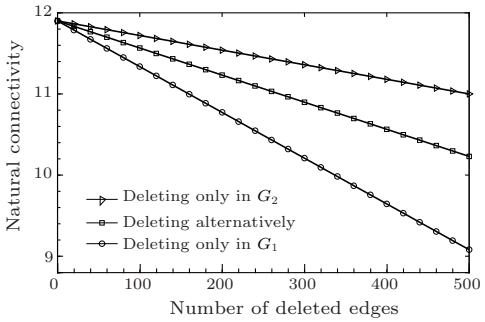

Fig. 1. The robustness measured by natural connectivity as a function of number of deleted edges for random strategy.

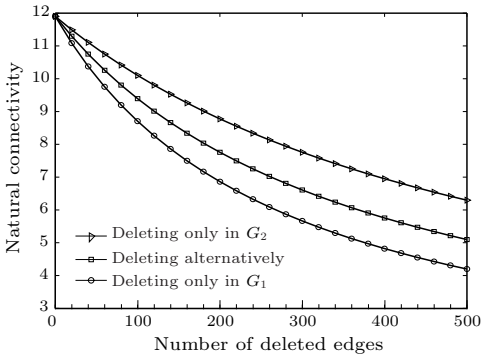

Fig. 2. The robustness measured by natural connectivity as a function of number of deleted edges for rich-rich strategy.

Next, to further compare the (holistic) natural connectivity $\bar{\lambda}$ with local natural connectivity, we consider an induced subgraph G_1 of G. Suppose that G_1 contains N_1 vertices and it is unnecessarily to be connected. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{N_1}$ be the eigenvalues of G_1 and $\bar{\lambda_1}$ the natural connectivity. Hence, the interlacing theorem for eigenvalues[26] implies

$$\lambda_i \geq \mu_i \geq \lambda_{N-N_1+i}, \quad 1 \leq i \leq N_1. \tag{7}$$

From Eqs. (3) and (7) we derive

$$\ln\left(\frac{1}{N_1}\sum_{i=1}^{N_1} e^{\lambda_{N-N_1+i}}\right) \leq \bar{\lambda_1} = \ln\left(\frac{1}{N_1}\sum_{i=1}^{N_1} e^{\mu_i}\right) \leq \ln\left(\frac{1}{N_1}\sum_{i=1}^{N_1} e^{\lambda_i}\right). \tag{8}$$

Note that from the definition, $\bar{\lambda}$ also satisfies the same bounds in Eq. (8). Consequently, we obtain

$$|\bar{\lambda} - \bar{\lambda_1}| \leq \ln\left(\frac{1}{N_1}\sum_{i=1}^{N_1} e^{\lambda_i}\right) - \ln\left(\frac{1}{N_1}\sum_{i=1}^{N_1} e^{\lambda_{N-N_1+i}}\right) = \ln\left(\frac{\sum_{i=1}^{N_1} e^{\lambda_i}}{\sum_{i=1}^{N_1} e^{\lambda_{N-N_1+i}}}\right), \tag{9}$$

which characterizes the dependency of local robustness to holistic robustness in complex networks.

For classical random graphs $G_{N,p}$,[32] the authors in Ref. [22] have shown that the natural connectivity

$$\bar{\lambda} = Np - \ln N + o(1) \tag{10}$$

holds almost surely when $N^{\varepsilon-1}\ln N < p < 1 - N^{\varepsilon-1}\ln N$ and $0 < \varepsilon < 1$. Hence, for a subgraph $G_1 = G_{N_1,p}$, by applying Eq. (9), the following inequality holds almost surely,

$$0 \leq \bar{\lambda} - \bar{\lambda_1} = (N - N_1)p - \ln\left(\frac{N}{N_1}\right) + o(1) \leq \ln\left(\frac{\sum_{i=1}^{N_1} e^{\lambda_i}}{\sum_{i=1}^{N_1} e^{\lambda_{N-N_1+i}}}\right), \tag{11}$$

when $N_1^{\varepsilon-1}\ln N_1 < p < 1 - N_1^{\varepsilon-1}\ln N_1$ and $0 < \varepsilon < 1$.

In summary, we have studied the local natural connectivity as a spectral measure of robustness in complex networks. We decompose the natural connectivity of a network as local natural connectivity of its connected components and quantify their contributions to the robustness of the whole network. The analytical results agree well with the numerical results. By using the eigenvalue interlacing inequality, we compare the natural connectivity of a network with that of an arbitrary induced subgraph. A new inequality involving eigenvalues of Erdös-Rényi random graph is derived as an application of our result. The natural connectivity has an acute discrimination in measuring the robustness of complex networks and rich information about the topological and dynamical properties can be extracted from the holistic/local spectral analysis of complex networks.

The author thanks an anonymous referee for reading the paper carefully and for suggesting the inclusion of a quantitative verification which is presented after Eq. (6).

References

[1] Albert R and Barabási A L 2002 *Rev. Mod. Phys.* **74** 47
[2] Amaral L A N and Uzzi B 2007 *Management Sci.* **53** 1033
[3] Boccaletti S, Latora V, Moreno Y, Chavez M and Hwang D U 2006 *Phys. Rep.* **424** 175
[4] Newman M E J 2003 *SIAM Rev.* **45** 167
[5] Albert R, Jeong H and Barabási A L 2000 *Nature* **406** 378
[6] Bollobás B and Riordan O 2004 *Internet Math.* **1** 1

[7] Chi L P, Yang C B and Cai X 2006 *Chin. Phys. Lett.* **23** 263
[8] Hu B, Li F and Zhou H S 2009 *Chin. Phys. Lett.* **26** 128901
[9] Wang J W and Rong L L 2008 *Chin. Phys. Lett.* **25** 3826
[10] Battilotti S 2008 *Automatica* **44** 348
[11] Shang Y 2010 *Int. J. Math. Anal.* **4** 1205
[12] Shang Y 2010 *Chin. Phys.* B **19** 070201
[13] Frank H and Frisch I 1970 *IEEE Trans. Commun. Tech.* **18** 567
[14] Esfahanian A H and Hakimi S 1988 *Inform. Process. Lett.* **27** 195
[15] Krishnamoorth M and Krishnamirthy B 1987 *Comput. Math. Appl.* **13** 577
[16] Chvátal V 1973 *Discrete Math.* **5** 215
[17] Jung H 1978 *J. Comb. Theor.* B **24** 125
[18] Mohar B 1989 *J. Comb. Theor.* B **47** 274
[19] Barrat A, Barthélemy M and Vespignani A 2008 *Dynamical Process on Complex Networks* (Cambridge: Cambridge University Press) p1
[20] Fiedler M 1973 *Czech. Math. J.* **23** 298
[21] Wu J, Barahona M, Tan Y J and Deng H Z 2010 *Chin. Phys. Lett.* **27** 078902
[22] Wu J, Barahona M, Tan Y J and Deng H Z 2010 arXiv:1009.3430
[23] Wu J, Barahona M, Tan Y J and Deng H Z 2009 arXiv:0912.2144
[24] Wu J, Deng H Z and Tan Y J 2010 *The Third IEEE International Conference on Computer Science and Information Technology* (Chengdu China 9–11 July 2010) p 50
[25] Shang Y 2011 *J. Phys. A: Math. Theor.* **44** 075003
[26] Haemers W H 1995 *Lin. Algebra Appl.* **227** 593
[27] Estrada E 2000 *Chem. Phys. Lett.* **319** 713
[28] Estrada E and Rodríguez-Velázquez J A 2005 *Phys. Rev.* E **72** 046105
[29] Estrada E and Rodríguez-Velázquez J A 2005 *Phys. Rev.* E **71** 056103
[30] Barabási A L and Albert R 1999 *Science* **286** 509
[31] Cohen R, Erez K, ben-Avraham D and Havlin S 2001 *Phys. Rev. Lett.* **86** 3682
[32] Bollobás B 2001 *Random Graphs* (Cambridge: Cambridge University) p 1