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Transient Growth of Perturbation Energy in the Taylor–Couette Problem with Radial Flow *

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Transient growth of perturbation energy in the Taylor–Couette problem with radial flow is investigated. The effects of radial flow on transient growth and structure of the optimal perturbation are mainly considered. For the wide gap case, strong radial flow, either inward or outward, shifts the peak of the amplitude of optimal perturbation towards the outer cylinder and the lift-up mechanism cannot be observed. However, for the narrow gap case, the optimal perturbation is almost unaffected by the radial flow and the lift-up mechanism still exists.

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The Taylor–Couette problem of flow between two independently rotating concentric cylinders has been a theoretical, numerical and experimental benchmark problem of fluid dynamics for more than eighty years.[1−3] Linear stability analysis is successful in explaining the onset of axisymmetric vortices with remarkable accuracy. However, Coles[4] and Van Atta[5] in the 1960s and later Hegseth *et al.*[6]in the 1980s reported the phenomena of sudden transition to spiral turbulence in the linearly stable region. This subcritical transition cannot be explained by eigenvalue analysis of linear stability theory. Instead, it may be associated with considerable transient growth of very small perturbations. Hristova *et al.*[7,8] studied energy transient growth of axisymmetric perturbations in the Taylor–Couette problem with fixed axial periodicity and counter-rotating cylinders. Transient growth was found to be strongly affected by curvature and the greatest transient growth is achieved on linear stability boundary, if it exists, or between linear stability boundary and narrow gap limit. In addition, a simultaneous work was provided by Meseguer[9] for a particular small gap case. Significant transient growth was found in the stable regime of counter-rotation where subcritical transition was observed in experiments carried out in a narrow gap apparatus.

The stability of Taylor–Couette flow can be altered when an additional flow, either axial[10,11] or radial,[12] is superimposed. These situations are often encountered in applications to rotating components in modern machinery.[13] In this Letter, we focus on how radial flow in the annulus affects the stability of the Taylor vortex flow. This type of flow often occurs in the process of dynamic filtration.[14] Some studies[15,16] by linear stability analysis have revealed that radially inward flow and strong outward flow have a stabilizing effect on the Taylor vortex flow, while weak outward flow has a destabilizing effect. Moreover, radially inward flow shifts the Taylor vortices toward the inner cylinder, while radially outward flow shifts the Taylor vortices toward the outer cylinder. Recent numerical simulation confirmed the predictions of radial flow effects on stability of the Taylor vortex flow.[17]

Although great efforts have been put into linear stability analysis on radial flow effect, no information is given about its impact on the transient growth of perturbations in the Taylor–Couette flow. In this study, our focus is on the transient evolution in the regime of linear stability. Energy transient growth of perturbation is investigated for the Taylor–Couette flow with radial flow. Two radius ratios (inner cylinder radius to outer cylinder radius) of 0.5 and 0.99 are studied, corresponding with the work of Hristova *et al.*[7]

We consider an incompressible fluid of kinematic viscosity ν and density ρ, which is contained between two concentric rotating cylinders whose radii and angular velocities are r_i^*, r_o^* and Ω_i, Ω_o, respectively. Following the analytical formulation by Bahl[18] and using $d = r_o^* - r_i^*$, $\Omega_i r_i^*$, $\Omega_i^2 r_i^{*2}$ as characteristic scales for space, velocity and the reduced pressure p/ρ, respectively, the dimensionless form of the basic velocity profile in cylindrical coordinates (r, θ, z) is

$$\begin{aligned} U(r) &= \frac{\alpha}{Ta}\frac{1}{r}, \\ V(r) &= \frac{(\mu-\eta^2)(1-\eta)^{\alpha+1}}{(1-\eta^{\alpha+2})\eta} r^{\alpha+1} + \frac{(1-\mu\eta^{\alpha})\eta}{(1-\eta^{\alpha+2})(1-\eta)}\frac{1}{r}, \\ W &= 0, \quad r_i \le r \le r_o, \end{aligned} \tag{1}$$

where $\eta = r_i^*/r_o^*$ is the radius ratio, $r_i = \eta/(1-\eta)$ and $r_o = 1/(1-\eta)$ are the dimensionless inner and outer radii. The parameters $\alpha = u_i^* r_i^*/\nu$, $Ta = \Omega_i r_i^* d/\nu$ and $\mu = \Omega_o/\Omega_i$ denote the radial Reynolds number, the Taylor number and the angular velocity ratio, respectively. For our analysis, the mean flow is supposed to be perturbed by a disturbance of velocity $(u, v, w)e^{i(m\theta+kz)}$ and pressure $pe^{i(m\theta+kz)}$, where k and m are the axial and azimuthal wave numbers, respectively. The evolution equation in terms of the

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complex amplitudes $\boldsymbol{u} = (u, v, w)$ is written symbolically as

$$\frac{\partial \boldsymbol{u}}{\partial t} = -i\mathcal{L}\boldsymbol{u}, \tag{2}$$

where $\mathcal{L}$ is a linear ordinary differential operator. Radial derivatives in the operator are discretized by Chebyshev spectral collocation method. The boundary conditions are such that the perturbation of velocity vanishes on the walls. As a measure of growth of arbitrary initial disturbance, we use the energy norm defined by

$$E(\boldsymbol{u}) = ||\boldsymbol{u}||_E^2 = \frac{1}{2}\int_{r_i}^{r_o} \left(|u|^2 + |v|^2 + |w|^2\right) r dr. \tag{3}$$

The maximal energy growth for $\boldsymbol{u}$ evolving according to Eq. (2) is defined by maximizing the ratio between the energy norm of the perturbation at time t and its initial norm

$$G(t) = \sup_{u(0)\neq 0} \frac{\|\boldsymbol{u}(t)\|_E^2}{\|\boldsymbol{u}(0)\|_E^2}, \tag{4}$$

which is calculated by the method described in chapter 4 of Ref. [19]. The optimal energy growth in time is defined as $G_O = G(t_O) = \sup_{t\geq 0} G(t)$, where t_O is the corresponding optimal time. The optimal perturbation, denoted by $\boldsymbol{u}(0)$, is the normalized initial condition which achieves the maximum (sup) in (4) for $t = t_O$.

In order to validate our numerical method and computational codes, eigenvalue analysis on Taylor–Couette flow with radial flow and transient growth analysis on Taylor–Couette flow without radial flow have been carried out. Results are compared well with those formerly provided.[8,15]

Our target is to study the effect of radial flow on transient growth in Taylor–Couette flow. Therefore a wide range of the radial Reynolds number is investigated, varying from -30 to 30. The other parameters are fixed at certain values as follows: $m = 0$, $k = \pi$ and $\mu = -1$. The choice of the parameters is similar to Hristova *et al.*[7] for the study of transient growth in Taylor–Couette problem without radial flow. Other values of k are also considered and the largest transient energy growth has been found at about $k = \pi$, especially for narrow gap case with significant transient growth. Note that their Reynolds number Re equals to our Taylor number Ta and their axial wavenumber β and k are related by $k = 2\beta$.

We begin by presenting the optimal growth as functions of α and Ta. Figure 1 shows the contours of constant optimal growth G_O. Inside the shaded region, the flow is linearly unstable to axisymmetric perturbations. The boundary of this region gives the critical Taylor number $Ta_c(\alpha)$ as a function of radial Reynolds number α. For a wide gap case ($\eta = 0.5$), varying the radial Reynolds number has a large effect on the critical Taylor number; while for a narrow gap case ($\eta = 0.99$), the radial Reynolds number has a relatively small effect. Thus the observation about the critical Ta here is consistent with the published results with fixed axial periodicity.[15] Furthermore, radial flow has a stronger effect on the optimal energy growth for a wider annular gap. For $\eta = 0.5$, at fixed Ta, G_O decreases rapidly with α, attains a minimum near $\alpha = 0$ and then increases again; while for $\eta = 0.99$, G_O changes quite slowly with α.

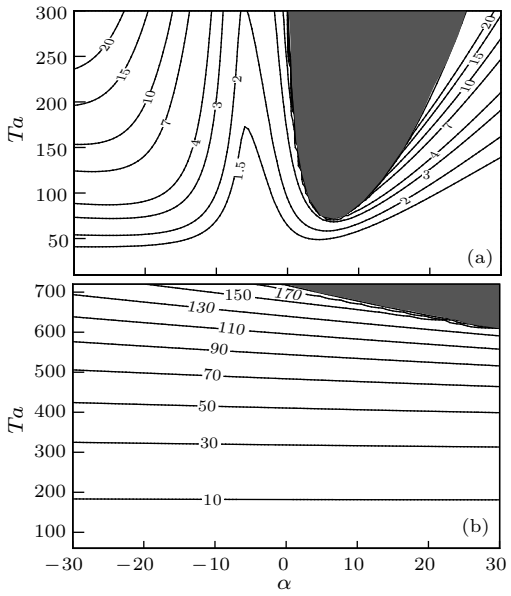

Fig. 1. Constant optimal growth G_O in the (α, Ta) plane. (a) $\eta = 0.5$. (b) $\eta = 0.99$.

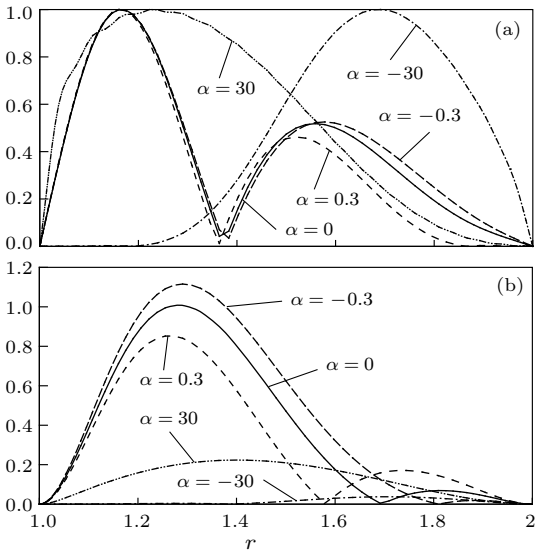

Fig. 2. The radial distribution of the modulus of the amplitude of the optimal perturbation with $\eta = 0.5$, $Ta = 250$. (a) The azimuthal direction. (b) The radial direction.

To better understand the effect of radial flow on the optimal transient growth of disturbances developing in the Taylor–Couette flow, the radial distribution of the modulus (the square root of sum of

real and imaginary parts squared) of the amplitude of the optimal perturbed azimuthal velocity is plotted in Fig. 2(a) for a wide gap case. In the case of no radial flow, the multi-peak shape of the curve is different from that of the least stable eigenmode, which features a single extreme.[15] Furthermore, all peaks are not located at the center of the annular gap. For radially inward flow the lower peak is shifted toward the outer cylinder and the larger one is eventually reduced to zero; while the larger peak for radially outflow is also shifted toward the outer cylinder by strong outward flow and the lower one is dissipated. The shifting of the larger peak is also evident for the amplitude of the perturbed radial velocity, as shown in Fig. 2(b). It should be noted that strong radial flow, inward or outward, would result in a great suppression of the radial perturbation. For the narrow gap case, the radial location of the peaks of the optimal perturbation remains nearly unaffected by radial flow, as seen in Fig. 3. Further, the amplitude of the perturbed radial velocity is more than eight times that of the azimuthal velocity. Compared with radially outward flow, radially inward flow suppresses the magnitude of the lower peak of the amplitude of the azimuthal perturbation slightly and leads to a greater enhancement of the radial perturbation.

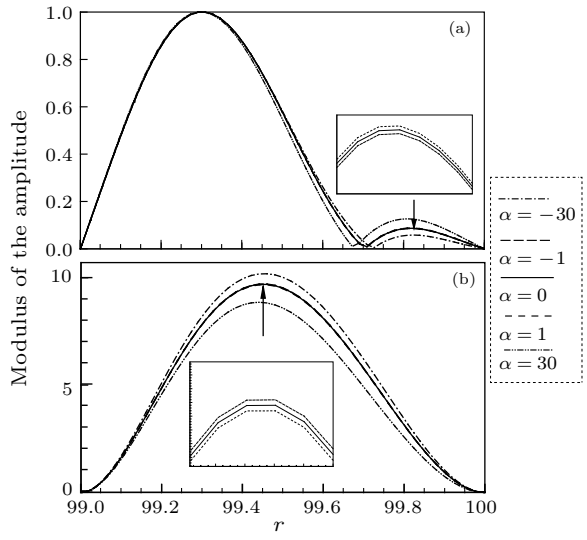

Fig. 3. The radial distribution of the modulus of the amplitude of the optimal perturbation with $\eta = 0.99$, $Ta = 600$. (a) The azimuthal direction. (b) The radial direction.

Clearly, radial flow shifts the position of the vortex within the annular gap between the cylinders for the wide gap case. The shift in the radial position of the vortex is quite evident in the perturbed radial-axial velocity vector plots shown in Fig. 4, where the contours of azimuthal velocity are also presented. In the case of no radial flow, the meridional velocity field consists of two arrays of vortices, a larger and stronger array lies near the inner cylinder with larger values of azimuthal velocity and a smaller and weaker array near the outer cylinder with smaller values of azimuthal velocity, being consistent with the results of Hristova *et al.*[7] When a strong outward flow is imposed, only one array of vortices appears in the optimal perturbation, whose centers are near the inner cylinder; while strong inward flow is imposed, two arrays of vortices are observed. The stronger vortices are located near the outer cylinder and the weaker vortices are located near the inner cylinder. The velocity vector and contours of the perturbed azimuthal velocity in the meridional plane for the narrow gap case are plotted in Fig. 5. The vector fields corresponding to different radial Reynolds number are quite similar except for some phase shift in the axial direction.

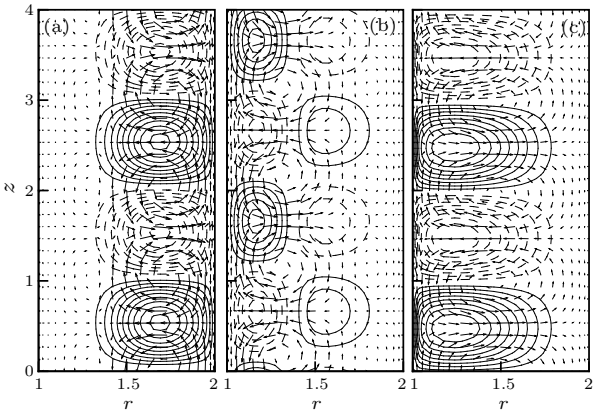

Fig. 4. The perturbed radial-axial velocity vector plots and the contours of azimuthal velocity in the meridional plane with $\eta = 0.5$, $Ta = 250$. (a) $\alpha = -30$. (b) $\alpha = 0$. and (c) $\alpha = 30$.

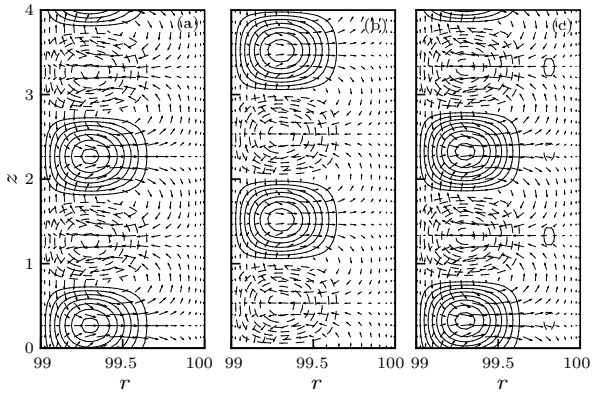

Fig. 5. The perturbed radial-axial velocity vector plots and the contours of azimuthal velocity in the meridional plane with $\eta = 0.99$, $Ta = 600$. (a) $\alpha = -30$. (b) $\alpha = 0$. and (c) $\alpha = 30$.

Finally, we show the evolution in time of the energies in the meridional components E_{rz} and in the azimuthal component E_θ starting from the optimal input $u(0)$. For the wide gap case, in the case of no radial flow, the optimal perturbation $u(0)$ is concentrated primarily in the meridional components ($E_{rz} \gg E_\theta$), as shown in Fig. 6(a). As time increases, E_{rz} decreases rapidly and E_θ increases. That is, the energy of the meridional components is transferred

to the azimuthal direction of the flow by the lift-up mechanism. This phenomenon results in the formation of azimuthal streaks. Nevertheless, while strong radial flow is considered ($|\alpha| = 30$), the situation is quite different. Initially, the flow field is dominated by the azimuthal velocity ($E_\theta \gg E_{rz}$). With time, all the three velocity components increase, thus the perturbation energies E_{rz} and E_θ are together increased until a maximum is reached. Subsequently, both of them decay as predicted by the linear stability analysis. These results would imply that for the wide gap case, when strong radial flow is added to the Taylor–Couette flow, the lift-up mechanism is not present and the perturbation energy growth is totally due to the energy transformation from the basic flow. Additionally, it should be noted that inward and outward radial flows may have different effects on the process of transient growth. For the parameters herein, radially inward flow leads to larger perturbation energy in the meridional plane than the azimuthal direction eventually; while for outflow, the opposite is true. Figure 6(b) plots the E_{rz} and E_θ curves corresponding to three values of α for the narrow gap case. Obviously, strong radial flow, inward or outward, does not change the fact that lift-up mechanism is responsible for the short-time energy growth in asymptotically stable flow regimes. This is in contrast to the wide gap case.

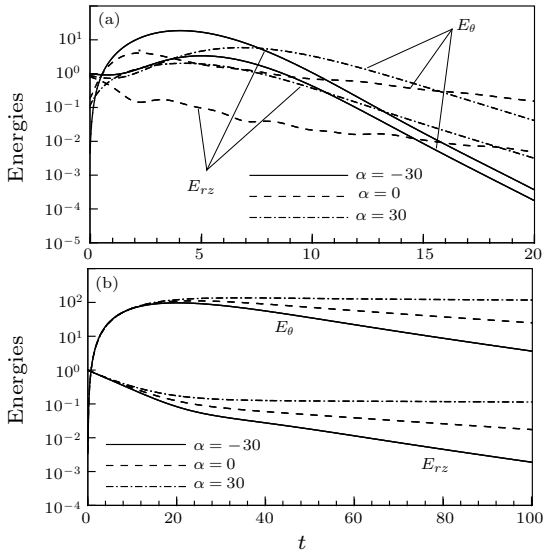

Fig. 6. Time evolution of the energies in the meridional components E_{rz} and in the azimuthal component E_θ of the optimal perturbation. (a) $\eta = 0.5$, $Ta = 250$. (b) $\eta = 0.99$, $Ta = 600$.

In conclusion, the transient growth characteristics of the Taylor–Couette flow with radial flow has been investigated for two values of η, i.e., the wide gap case with $\eta = 0.5$ and the narrow gap case with $\eta = 0.99$. We find that the effects of radial flow on the energy transient growth and flow structure of the optimal perturbations are quite different for the two cases. For the wide gap case, strong radial flow, either inward or outward, shifts the peak of the amplitude of optimal perturbation towards the outer cylinder. Note that the effect of radial flow on the structure of the least stable normal mode perturbation is very different, i.e., the peak of the amplitude of the perturbation moves in the same direction as radial flow. Furthermore, the lift-up mechanism cannot be observed when strong radial flow is imposed, as the perturbation field features simultaneous energy growth in meridional plane and azimuthal direction. However, for the narrow gap case, the optimal perturbation is almost unaffected by the radial flow and no matter how strong radial flow is considered, the lift-up mechanism is acceptable to explain the dynamic process of transient growth, since the transfer of perturbation energy from azimuthal vorticity to azimuthal velocity can still be seen clearly.

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