CHIN. PHYS. LETT. Vol. 28, No. 12 (2011) 121401

Flavor State of the Neutrino: Conditions for a Consistent Definition

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(Received 13 July 2011)

Similar to Blasone et al. [Phys. Rev. D 72 (2005) 013003] by examining the expectation value of the flavor charge under the normalized flavor state of the neutrino, we demonstrate that the introduction of the flavor state is consistent with the flavor charge only when the conditions, i.e. (A) $|\boldsymbol{p}| = 0$ (the low energy case), or $|\boldsymbol{p}|/m_i \gg 1$ (the relativistic case) and (B) the flavor state should be defined of the Pontecorvo form $|\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle$ or equivalently except a global phase, are satisfied. The root of the issue lies in the structure of the flavor charge operator. The diagonalization of the flavor charge operator with the integer eigenvalue can be realized through the Bogoliubov–Valatin transformation when condition A is satisfied. The eigenstate of the diagonalized flavor charge is of the Pontecorvo form under condition B.

PACS: 14.60.Pq, 03.70.+k, 03.65.−w DOI:10.1088/0256-307X/28/12/121401

Although neutrinos were discovered about half a century ago,[1] the nature of the neutrino flavor state is still unclear. Especially in the discussion of the neutrino oscillation promoted by a series of neutrino experiments, the controversy on the neutrino flavor state does not diminish but grows.[2,3] Specifically, Blasone *et al.*[4,5] considered the flavor state as the result of the quantized neutrino flavor field. Giunti[6,7] considered that the Fock space of the flavor neutrinos is not physical. He pointed out that the Pontecorvo state $|\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle$ is not the flavor state or weak state and proposed that the flavor state is dependent on the production process or the detection process. On the basis of the conservation of the energy-momentum, Cohen *et al.*[8] considered the entanglement between the massive neutrino and the recoiling particle. However, Akhmedov and Smirnov[9] considered that the entanglement and disentanglement are unnecessary under the realistic condition. Despite these modifications to the flavor state or the produced state, the picture of the flavor state is still unclear. In this Letter, we discuss the conditions for a consistent definition of the flavor state and the root of the issue.

As a self-consistent interpretation, the expectation value of the flavor charge under the flavor state of the N neutrinos is N

$$\frac{\langle\alpha|Q_\alpha|\alpha\rangle}{\langle\alpha|\alpha\rangle} = N, \quad (N = 1, 2, 3, 4, \ldots) \tag{1}$$

where $|\alpha\rangle$ is the normalized flavor state of N neutrinos and Q_α is the corresponding flavor charge operator. We will examine whether this self-consistent condition could be realized. For simplicity, we consider the two-flavor single particle case ($N = 1$),

$$\nu_e(\boldsymbol{x}) = \cos\theta\nu_1(\boldsymbol{x}) + \sin\theta\nu_2(\boldsymbol{x}), \tag{2}$$

$$\nu_\mu(\boldsymbol{x}) = -\sin\theta\nu_1(\boldsymbol{x}) + \cos\theta\nu_2(\boldsymbol{x}), \tag{3}$$

where $\nu_e(\boldsymbol{x})$ and $\nu_\mu(\boldsymbol{x})$ are the quantized fields of the flavor neutrinos. The case of three-flavor multi-particles can be dealt with in a similar way and the result is invariant. We first consider the flavor state of the single particle with a fixed momentum. The Pontecorvo flavor states with the momentum $\boldsymbol{p}$ read

$$|\nu_e^s(\boldsymbol{p})\rangle = \cos\theta|\nu_1^s(\boldsymbol{p})\rangle + \sin\theta|\nu_2^s(\boldsymbol{p})\rangle, \tag{4}$$

$$|\nu_\mu^s(\boldsymbol{p})\rangle = -\sin\theta|\nu_1^s(\boldsymbol{p})\rangle + \cos\theta|\nu_2^s(\boldsymbol{p})\rangle, \tag{5}$$

where s denotes the helicity of the neutrino. The massive state $|\nu_i^s(\boldsymbol{p})\rangle = c_i^{s+}(\boldsymbol{p})|0\rangle$ $(i = 1, 2)$ follows the normalization condition $\langle\nu_i^s(\boldsymbol{p})|\nu_j^r(\boldsymbol{q})\rangle = \delta_{sr}\delta_{ij}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q})$. Following the conventional quantization procedure, the massive field is of the form

$$\nu_i(x) = \sum_{r=1}^{2}\int \frac{1}{(2\pi)^{\frac{3}{2}}} d^3\boldsymbol{p}\sqrt{\frac{m_i}{E_{i\boldsymbol{p}}}}[u_{ir}(\boldsymbol{p})c_{ir}(\boldsymbol{p})e^{ipx} + \nu_{ir}(\boldsymbol{p})d_{ir}^+(\boldsymbol{p})e^{-ipx}], \quad (i = 1, 2) \tag{6}$$

where $E_{i\boldsymbol{p}} = \sqrt{m_i^2 + \boldsymbol{p}^2}$. The flavor charge operator reads[10]

$$Q_{\nu e} = \int d^3\boldsymbol{x} : \nu_e^+(x)\nu_e(x) :, \tag{7}$$

where : : is the normal ordering defined as $: cc^+ := -c^+c$. With the help of the anti-commutation relations $\{c_{ir}(\boldsymbol{p}), c_{js}^+(\boldsymbol{q})\} = \delta_{sr}\delta_{ij}\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q})$, we obtain the expectation value of the flavor charge

$$\frac{\langle\nu_e^s(\boldsymbol{p})|Q_{\nu e}(t = 0)|\nu_e^s(\boldsymbol{p})\rangle}{\langle\nu_e^s(\boldsymbol{p})|\nu_e^s(\boldsymbol{p})\rangle} = \cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta g(E_{i\boldsymbol{p}}, m_i), \tag{8}$$

where $g(E_{i\boldsymbol{p}}, m_i)$ is of the form[11]

$$g(E_{i\boldsymbol{p}}, m_i) = \sqrt{\frac{1}{2}\Big(1 + \frac{\boldsymbol{p}^2 + m_1m_2}{E_{1\boldsymbol{p}}E_{2\boldsymbol{p}}}\Big)}. \tag{9}$$

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Then $\boldsymbol{p}^2$ can be parameterized as $\boldsymbol{p}^2 = \alpha^2 m_1^2 = \beta^2 m_2^2 (\alpha, \beta \geq 0)$. Thus $g^2(\alpha, \beta)$ is of the form

$$g^2(\alpha,\beta) = \frac{1}{2}\Big(1 + \frac{\alpha\beta + 1}{\sqrt{\alpha^2+1}\sqrt{\beta^2+1}}\Big). \tag{10}$$

We discuss the function $g^2(\alpha, \beta)$ in detail as follows.

Case A: $\alpha = \beta = 0$, equivalently, $\boldsymbol{p} = 0$. We obtain $g^2(\alpha, \beta) = 1$, and the expectation value

$$\frac{\langle \nu_e^s(\boldsymbol{p})|Q_{\nu e}(t=0)|\nu_e^s(\boldsymbol{p})\rangle}{\langle \nu_e^s(\boldsymbol{p})|\nu_e^s(\boldsymbol{p})\rangle} = 1.$$

Case B: $\alpha = \beta > 0$, equivalently, $m_1 = m_2$, $|\boldsymbol{p}| > 0$. We obtain the same result as in the case A. However, the neutrino oscillation implies the premise requirement $m_2^2 - m_1^2 \neq 0$. Therefore, this case is precluded from our consideration.

Case C: $\alpha \to \infty$, $\beta \to \infty$, equivalently, $|\boldsymbol{p}|/m_i \gg 1$ $(i = 1, 2)$. We obtain $g^2(\alpha, \beta) \simeq 1$ and the expectation value

$$\frac{\langle \nu_e^s(\boldsymbol{p})|Q_{\nu e}(t=0)|\nu_e^s(\boldsymbol{p})\rangle}{\langle \nu_e^s(\boldsymbol{p})|\nu_e^s(\boldsymbol{p})\rangle} \simeq 1.$$

Case D: $0 < \alpha < \infty, 0 < \beta < \infty$, $\alpha \neq \beta$, equivalently, $m_1 \neq m_2$, $0 < \frac{|\boldsymbol{p}|}{m_i} < \infty$. We obtain $g^2(\alpha, \beta) < 1$ and the expectation value

$$\frac{\langle \nu_e^s(\boldsymbol{p})|Q_{\nu e}(t=0)|\nu_e^s(\boldsymbol{p})\rangle}{\langle \nu_e^s(\boldsymbol{p})|\nu_e^s(\boldsymbol{p})\rangle} = \eta \quad (0 < \eta < 1).$$

Therefore, in this universal case, the introduction of the Pontecorvo flavor state is not consistent with the flavor charge.

As a conclusion from the above discussion, the introduction of the Pontecorvo flavor state and flavor charge is consistent only when the condition $\boldsymbol{p} = 0$ or $|\boldsymbol{p}| \gg m_i (i = 1, 2)$ is satisfied. One may consider that this condition is dependent on the special definition of the flavor state. Thus we will examine the conclusion under the flavor sate of the more universal form.

As a more complete examination, we consider the flavor state of the single particle as the superposition of the momentum eigenstates,

$$|\nu_e^s\rangle = x\int d^3\boldsymbol{p} f_1(\boldsymbol{p})|\nu_1^s(\boldsymbol{p})\rangle + y\int d^3\boldsymbol{p} f_2(\boldsymbol{p})|\nu_2^s(\boldsymbol{p})\rangle, \tag{11}$$

where the coefficients satisfy the normalization condition, $|x|^2 + |y|^2 = 1$ and $\int d^3\boldsymbol{p} f_1^*(\boldsymbol{p}) f_1(\boldsymbol{p}) = \int d^3\boldsymbol{p} f_2^*(\boldsymbol{p}) f_2(\boldsymbol{p}) = 1$. With the help of the parameterized coefficients $x = \cos\theta' e^{i\theta_1}$ and $y = \sin\theta' e^{i\theta_2} (0 < \theta' < \frac{\pi}{2})$, we can obtain the conclusion that only when following conditions (or equivalent conditions) are satisfied

$$\int d^3\boldsymbol{p} f_1^*(\boldsymbol{p}) f_1(\boldsymbol{p}) g(E_{i\boldsymbol{p}}, m_i) = 1, \tag{12}$$

$$f_2(\boldsymbol{p}) = e^{i(\theta_1 - \theta_2)} f_1(\boldsymbol{p}), \tag{13}$$

$$\theta - \theta' = n\pi \quad (n = 0, \pm 1), \tag{14}$$

we can obtain the expectation value $\frac{\langle \nu_e^s|Q_{\nu e}(t=0)|\nu_e^s\rangle}{\langle \nu_e^s|\nu_e^s\rangle} = 1$. Condition (12) can be realized when the momentum distribution focuses on the low energy area $\boldsymbol{p} = 0$ or on the relativistic area $\frac{|\boldsymbol{p}|}{m_i} \gg 1$ $(i = 1, 2)$. Conditions (13) and (14) mean that the flavor state should be defined as

$$|\nu_e^s\rangle = e^{i\alpha}\int d^3\boldsymbol{p} f(\boldsymbol{p})(\cos\theta|\nu_1^s(\boldsymbol{p})\rangle + \sin\theta|\nu_2^s(\boldsymbol{p})\rangle). \tag{15}$$

Therefore, the conclusion under the superposed state is the same as the conclusion under the momentum eigenstate. The re-definition of the flavor state is limited by the requirement of the normalization and the requirement of the consistency.

As we know, the flavor state is the eigenstate of the flavor charge. The introduction of a consistent flavor state depends on whether the flavor charge operator can be diagonalized with the integer eigenvalue. The flavor charge of the neutrino is of the form

$$\begin{aligned} Q_{\nu e}(t) = \sum_{s=1}^{2}\int d^3\boldsymbol{p}\Big[& \cos^2\theta c_{1s}^+(\boldsymbol{p})c_{1s}(\boldsymbol{p}) \\ & + \sin^2\theta c_{2s}^+(\boldsymbol{p})c_{2s}(\boldsymbol{p}) \\ & + \sin\theta\cos\theta g(E_{i\boldsymbol{p}}, m_i)e^{-i(E_{1\boldsymbol{p}} - E_{2\boldsymbol{p}})t}c_{1s}^+(\boldsymbol{p})c_{2s}(\boldsymbol{p}) \\ & + \sin\theta\cos\theta g(E_{i\boldsymbol{p}}, m_i)e^{-i(E_{2\boldsymbol{p}} - E_{1\boldsymbol{p}})t}c_{2s}^+(\boldsymbol{p})c_{1s}(\boldsymbol{p}) \\ & - \cos^2\theta d_{1s}^+(\boldsymbol{p})d_{1s}(\boldsymbol{p}) - \sin^2\theta d_{2s}^+(\boldsymbol{p})d_{2s}(\boldsymbol{p}) \\ & - \sin\theta\cos\theta g(E_{i\boldsymbol{p}}, m_i)e^{-i(E_{1\boldsymbol{p}} - E_{2\boldsymbol{p}})t}d_{1s}^+(\boldsymbol{p})d_{2s}(\boldsymbol{p}) \\ & - \sin\theta\cos\theta g(E_{i\boldsymbol{p}}, m_i)e^{-i(E_{2\boldsymbol{p}} - E_{1\boldsymbol{p}})t}d_{2s}^+(\boldsymbol{p})d_{1s}(\boldsymbol{p})\Big], \end{aligned} \tag{16}$$

where c^+ is the production operator of a massive neutrino, and d^+ is the production operator of a massive anti-neutrino. There are no mixing items of operators of the neutrino and the anti-neutrino in the flavor charge. They are eliminated by the zero coefficients $\delta(\boldsymbol{p})f(\boldsymbol{p})$, where $f(\boldsymbol{p})$ is the odd function of the variable $\boldsymbol{p}$. The unitary transformation of the operator does not change its eigenvalue. Before considering the special transformation that diagonalizes the flavor charge operator, we first examine the eigen-equation of the quadratic form of the flavor charge

$$\begin{aligned} & (\cos^2\theta - q_{\nu e}(\boldsymbol{p}))(\sin^2\theta - q_{\nu e}(\boldsymbol{p})) \\ & - \cos^2\theta\sin^2\theta g^2(E_{i\boldsymbol{p}}, m_i) = 0. \end{aligned} \tag{17}$$

The eigenvalue is

$$q_{\nu e}(\boldsymbol{p}) = \frac{1 \pm \sqrt{1 - 4\cos^2\theta\sin^2\theta(1 - g^2(E_{i\boldsymbol{p}}, m_i))}}{2}. \tag{18}$$

Obviously, only when $g^2(E_{i\boldsymbol{p}}, m_i) = 1$, equivalently $\boldsymbol{p} = 0$, or $\frac{|\boldsymbol{p}|}{m_i} \gg 1$, we can obtain the physical eigenvalue $q_{\nu e}(\boldsymbol{p}) = 0, 1$. In this case the quadratic form of

the flavor charge can be converted to the square form

$$Q_{\nu e}(t) = \sum_{s=1}^{2} \int_{g(E_{i\boldsymbol{p}},m_i)=1} d^3\boldsymbol{p}[c_s^+(\boldsymbol{p},t)c_s(\boldsymbol{p},t) - d_s^+(\boldsymbol{p},t)d_s(\boldsymbol{p},t)], \tag{19}$$

where the transformation of the operators is

$$\begin{aligned} c_s(\boldsymbol{p},t) &= \cos\theta c_{1s}(\boldsymbol{p})e^{iE_{1\boldsymbol{p}}t} + \sin\theta c_{2s}(\boldsymbol{p})e^{iE_{2\boldsymbol{p}}t}, \\ d_s(\boldsymbol{p},t) &= \cos\theta d_{1s}(\boldsymbol{p})e^{iE_{1\boldsymbol{p}}t} + \sin\theta d_{2s}(\boldsymbol{p})e^{iE_{2\boldsymbol{p}}t}. \end{aligned} \tag{20}$$

Thus the flavor state can be derived as

$$\begin{aligned} |\nu_e^s(\boldsymbol{p},t)\rangle = c_s^+(\boldsymbol{p},t)|0\rangle = \ & \cos\theta c_{1s}^+(\boldsymbol{p})e^{-iE_{1\boldsymbol{p}}t}|0\rangle \\ & + \sin\theta c_{2s}^+(\boldsymbol{p})e^{-iE_{2\boldsymbol{p}}t}|0\rangle. \end{aligned} \tag{21}$$

When the condition $g^2(E_{i\boldsymbol{p}}, m_i) = 1$ is not satisfied, the eigenvalue of the diagonalized operator $q_{\nu e}(\boldsymbol{p})$ is not the integer. Thus the diagonalized operator can not be interpreted as the flavor charge with the momentum $\boldsymbol{p}$.

As has been discussed, only when following conditions are satisfied: (A) $|\boldsymbol{p}| = 0$ (the low energy case), or $\frac{|\boldsymbol{p}|}{m_i} \gg 1$ (the relativistic case); (B) the flavor state should be defined in the form $|\nu_e\rangle = \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle$ or equivalently except a global phase, the introduction of the flavor state is consistent with the eigenstate of the flavor charge. The direct root of the issue lies in the quadratic form of the flavor charge. In essence, as long as we introduce the flavor charge and deal with the massive field according the conventional quantization procedure, we will encounter extra conditions such as (A) and (B) for a consistent definition of the flavor state. To avoid these extra conditions, one may consider the neutrino oscillation in the quantum field theoretic frame without the introduction of the flavor state, such as in the Akhmedov and Kopp procedure.[12] As another possibility, when the quantization procedure of the massive field is modified, which means that the mass parameter is treated in a different way so that the factor $g(E_{i\boldsymbol{p}}, m_i)$ $(i = 1, 2)$ is eliminated, we can introduce the flavor state without the limitation to the momentum distribution.

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