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Vibrational and Stochastic Resonance in the FitzHugh–Nagumo Neural Model with Multiplicative and Additive Noise

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The vibrational resonance and stochastic resonance phenomena in the FitzHugh–Nagumo (FHN) neural model, driven by a high-frequency (HF) signal and a low-frequency (LF) signal and by coupled multiplicative and additive noises, is investigated. For the case that the frequency of the HF signal is much higher than that of the LF signal, under the adiabatic approximation condition, the expression of the signal-to-noise ratio (SNR) with respect to the LF signal is obtained. It is shown that the SNR is a non-monotonous function of the amplitude and frequency of the HF signal. In addition, the SNR varies non-monotonically with the increasing intensities of the multiplicative and additive noise as well as with the increasing system parameters of the FHN model. The influence of the coupling strength between the multiplicative and additive noises on the SNR is discussed.

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Stochastic resonance (SR) describes the phenomenon where an appropriate amount of noise is of constructive use in the sense that a weak signal becomes amplified upon harvesting the ambient noise in nonlinear systems.[1] Since its first discovery in the early eighties, SR has been observed in a great variety of systems pertaining to different disciplines such as physics, chemistry, engineering, biology and biomedical sciences.[1−4] The phenomenon vibrational resonance (VR) was named by Landa and McClintock.[5] They considered the effect of high-frequency (HF) force on the response of a bistable system to a low-frequency (LF) signal and found that the response can be optimized by an appropriate choice of the amplitude of the HF signal. Two-frequency signals are important for communication systems, since usually an LF signal modulates an HF carrier signal and is also an object of interest in other fields,[6,7] such as acoustics, neuroscience, laser physics, or engineering. The VR was observed in an experiment performed by means of an analog circuit.[8] The VR phenomenon in a damped quintic oscillator with monostable potential,[9] in a bistable potential subject to the action of a bichromatic force and additive noise, was also studied.[10]

The FitzHugh–Nagumo (FHN) model, proposed by Hodgkinan and Huxley,[11] also called the Vander Pol-Bonhoeffer model,[12] is a simple example of a two-dimensional excitable dynamics. A stochastic version of the FHN model was studied for the first time by Treutlein and Schulten.[13] Nonlinear dynamics of coupled FHN neurons subject to independent noise,[14] with a strong periodic driving,[15] in extended FHN system,[16] the effect of an HF signal on the FHN excitable model,[17] was investigated. However, only one signal, i.e., an HF signal was considered in the system. It is well known that a system is always disturbed simultaneously by external environmental perturbations (multiplicative noise) and intrinsic thermal fluctuations (additive noise), and there are certain situations in which the strong external perturbations can lead to some changes in the internal structure of the system.[18] Thus the VR phenomenon in FHN model with multiplicative and additive noises, driving simultaneously by an HF signal and an LF signal, needs to be investigated. In this Letter, we study the non-monotonous influence of noises and system parameters of the FHN model on the output signal-to-noise ratio (SNR) with respect to the LF signal.

We considering the FHN neural model[19]

$$\frac{dv}{dt} = v(a-v)(v-1) - w, \quad \frac{dw}{dt} = bv - rw. \tag{1}$$

In the context of neurophysiology, $v(t)$ denotes a fast variable representing the neuron membrane voltage and $w(t)$ the recovery variable. Parameter a $(0 < a < 1)$ is the threshold value, b and r are positive constants. The drift field for model (1) is given by

$$\begin{aligned} y_1(v,w) &= v(a-v)(v-1) - w, \\ y_2(v,w) &= bv - rw. \end{aligned} \tag{2}$$

One can see from Eq. (2) that the null cline of the deterministic dynamics of Eq. (2) is the line $v = rw/b$. By substitution on the right-hand side of Eq. (1), one can obtain the equation for the steady state,[20]

$$v(a-v)(v-1) - bv/r = 0. \tag{3}$$

In the present work, we take $r = 1$ for simplicity. For properly selected values of the parameters, Eq. (3) has three real roots, i.e., two stable states $v_{S1} = 0$, $v_{S2} = \frac{1}{2}[a+1+\sqrt{(a-1)^2-4b}]$ and one unstable state $v_{un} = \frac{1}{2}[a+1-\sqrt{(a-1)^2-4b}]$. Performing integration over line $v = rw/b$, one can obtain the potential function

$$V(v) = \frac{a+b}{2}v^2 - \frac{a+1}{3}v^3 + \frac{1}{4}v^4. \tag{4}$$

When one considers the external environmental fluctuations on the parameter b, one can rewrites it as $b + \xi(t)$. At the same time, the system is under an

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intrinsic thermal fluctuation $\eta(t)$ and external forcing $g(t)$, the dimensionless form of the one-dimension Langevin equation corresponding to potential function Eq. (4) can be written as

$$\begin{aligned}\frac{dv}{dt} &= v(a-v)(v-1) - bv - v\xi(t) + \eta(t) + g(t),\\ g(t) &= A_{\rm L}\sin(\Omega_{\rm L}t) + A_{\rm H}\sin(\Omega_{\rm H}t),\end{aligned} \tag{5}$$

where $A_{\rm H}$, $\Omega_{\rm H}$ and $A_{\rm L}$, $\Omega_{\rm L}$ denote the amplitudes and frequencies of the HF and LF signals, respectively. We assume that the HF frequency is much higher than the LF frequency, $\Omega_{\rm H} \gg \Omega_{\rm L}$. Here $\xi(t)$ and $\eta(t)$ are white multiplicative and additive noise terms, with zero mean and strengths D and P, respectively.

$$\langle \xi(t_1)\eta(t_2)\rangle = \langle \eta(t_1)\xi(t_2)\rangle = 2\lambda\sqrt{DP}\delta(t_1 - t_2). \tag{6}$$

Here λ is the coupling strength between the multiplicative and additive noises. We assume that the approximate solution of Eq. (5) has the form [7,8] $v(t) = u(t) + \psi(t, \Omega_{\rm H}t)$, where $u(t)$ is the slow motion component of the system response and $\psi(t, \Omega_{\rm H}t)$ is the fast motion response from the HF input, with periodic $T = 2\pi/\Omega_{\rm H}$, whose mean $\overline{\psi(t, \Omega\tau)} = \frac{1}{2\pi}\int_0^{2\pi}\psi(t,\tau)d\tau = 0$. Assuming that this periodic averaging of the HF signal does not change the characters of the multiplicative and additive noise, from Eqs. (5)–(6), it is possible to obtain the following evolution equations for the fast motion $\psi(t, \Omega_{\rm H}t)$:

$$\begin{aligned}\dot{\psi} =& -(a+b)\psi + (a+1)\psi^2 + 2(a+1)u(\psi - \overline{\psi})\\ & - (\psi^3 - \overline{\psi^3}) - 3u^2(\psi - \overline{\psi}) - 3u(\psi^2 - \overline{\psi^2})\\ & - \psi\xi(t) + \eta_1(t) + A_{\rm H}\cos(\Omega_{\rm H}t).\end{aligned} \tag{7}$$

Here $\eta_1(t)$, derived from $\eta(t)$, represents the noise term added to the HF motion component of Eq. (5). Since ψ is a fast changing force, we assume that $\dot{\psi} \gg \psi, \psi^2, \psi^3$. For $D \ll 1$, $P \ll 1$, the right-hand terms of Eq. (5) can be neglected except the term $A_{\rm H}\cos(\Omega_{\rm H}t)$, then one can obtain an approximate evolution equation for the fast part of Eq. (5), $\dot{\psi} = A_{\rm H}\cos(\Omega_{\rm H}t)$, whose solution is $\psi = \sin(\Omega_{\rm H}t)A_{\rm H}/\Omega_{\rm H}$. From Eqs. (5) and (6), the corresponding Fokker–Planck equation for the probability density $\rho(x,t)$ is

$$\frac{\partial}{\partial t}\rho(v,t) = -\frac{\partial}{\partial x}[B(v)\rho(v,t)] + \frac{\partial^2}{\partial x^2}[G(v)\rho(v,t)], \tag{8}$$

with $B(v) = v(a-v)(v-1) - bv + g(t) + Dv - \lambda\sqrt{DP}$, $G(v) = Dv^2 - 2\lambda\sqrt{DP}v + P$. Taking the advantage of Eq. (8), by using the process $u(t)$ instead of $v(t)$, the Fokker–Planck equation (8) can be rewritten as

$$\begin{aligned}&\frac{\partial}{\partial t}\rho(u, \Omega_{\rm H}, t)\\ &= -\frac{\partial}{\partial x}\Big[B\Big(u + \frac{A_{\rm H}}{\Omega_{\rm H}}\cos(\Omega_{\rm H}t)\Big)\rho(u, \Omega_{\rm H}, t)\Big]\\ &\quad + \frac{\partial^2}{\partial x^2}\Big[G\Big(u + \frac{A_{\rm H}}{\Omega_{\rm H}}\cos(\Omega_{\rm H}t)\Big)\rho(u, \Omega_{\rm H}, t)\Big].\end{aligned} \tag{9}$$

From Eq. (9), the probability density $\rho(u, \Omega_{\rm H}, t)$ is a periodic function of the periodic of the HF signal. Since our aim is to investigate the response with respect to the LF signal, we define a frequency average of an arbitrary function $f(u, \Omega_{\rm H}, t)$ as

$$\bar{f}(u,t) = \overline{f(u, \Omega_{\rm H}, t)} := \frac{1}{2\pi}\int_0^{2\pi/T} d\Omega_{\rm H} f(u, \Omega_{\rm H}, t). \tag{10}$$

For any frequency $\Omega_{\rm H} \in [0, 2\pi/T]$, it is expected that as $t \to \infty$, $\rho(u, \Omega_{\rm H}, t)$ depends on the LF signal and is almost independent of the high-frequency $\Omega_{\rm H}$, thus one can approximate $\rho(u, \Omega_{\rm H}, t) \approx \bar{\rho}(u,t)$ and the decoupling approximation $\overline{f(u, \Omega_{\rm H}, t)\rho(u, \Omega_{\rm H}, t)} \approx \bar{f}(u,t)\bar{\rho}(u,t)$ also holds. Applying the frequency average in Eq. (9) and using the decoupling approximation, we can obtain

$$\frac{\partial}{\partial t}\bar{\rho}(u,t) = -\frac{\partial}{\partial x}[\bar{B}(u))\bar{\rho}(u,t)] + \frac{\partial^2}{\partial x^2}[\bar{G}(u)\bar{\rho}(u,t)], \tag{11}$$

with

$$\bar{B}(u) = -a_2 u - a_1 u^2 - u^3 - a_3 + A_{\rm L}\cos(\Omega_{\rm L}t), \tag{12}$$

$$\bar{G}(u) = Du^2 + P + \frac{DA_{\rm H}^2}{2\Omega_{\rm H}^2} - 2\lambda\sqrt{DP}u, \tag{13}$$

$$a_1 = -(a+1), \quad a_2 = a + b + \frac{3A_{\rm H}^2}{2\Omega_{\rm H}^2} - D,$$

$$a_3 = -\frac{(a+1)A_{\rm H}^2}{2\Omega_{\rm H}^2} + \lambda\sqrt{DP}. \tag{14}$$

Under condition $p < 0$, $p^3 + q^2 < 0$, $p = \frac{a_2}{3} - \frac{a_1^2}{9}$, $q = \frac{a_1^3}{27} - \frac{a_1 a_2}{6} + \frac{a_3}{2}$, Eq. (12) have three states, i.e., the LF responding system of Eq. (5) has two stable states $u_{S1} = 2\sqrt{|p|}\cos(\frac{\phi}{3}) + \frac{a+1}{3}$, $u_{S2} = -2\sqrt{|p|}\cos(\frac{\phi-\pi}{3}) + \frac{a+1}{3}$ and one unstable state $u_{un} = -2\sqrt{|p|}\cos(\frac{\phi+\pi}{3}) + \frac{a+1}{3}$, with $\phi = \arccos(q|p|^{-3/2})$. In the following, we properly select the parameters in order that this condition is satisfied. Assuming that the frequency of the LF signal is smaller than the rate of the intrawell relaxation, there is enough time for the system to reach the equilibrium during the period of LF signal.[21] The solution of the corresponding Fokker–Planck equation is given by

$$\begin{aligned}\bar{\rho}_{st}(u) =& \frac{M}{\sqrt{\bar{G}(u)}}\exp\Big[\int^y \frac{\bar{B}(u)}{\bar{G}(u)}\Big]\\ =& \frac{M}{\sqrt{\bar{G}(u)}}\exp\Big[-\frac{\Phi(u)}{D}\Big],\end{aligned} \tag{15}$$

where M is the normalization constant, $\Phi(u)$ is the rectified potential function

$$\Phi(u) = \int_{-\infty}^{u}\frac{D\bar{B}(y)}{\bar{G}(y)}dy = \frac{u^2}{2} - (a+1+2\lambda\sqrt{P/D})u - \frac{m}{2}\ln\Big[(u - \lambda\sqrt{P/D})^2 + \frac{A_{\rm H}^2}{2\Omega_{\rm H}^2} + \frac{P(1-\lambda^2)}{D}\Big]$$

$$-\frac{n}{\sqrt{\frac{A_{\rm H}^2}{2\Omega_{\rm H}^2}+\frac{P(1-\lambda^2)}{D}}}\arctan\left(\frac{u}{\sqrt{\frac{A_{\rm H}^2}{2\Omega_{\rm H}^2}+\frac{P(1-\lambda^2)}{D}}}\right)$$
$$-\frac{A_{\rm L}\cos(\Omega_{\rm L}t)}{\sqrt{\frac{A_{\rm H}^2}{2\Omega_{\rm H}^2}+\frac{P(1-\lambda^2)}{D}}}\arctan\left(\frac{u}{\sqrt{\frac{A_{\rm H}^2}{2\Omega_{\rm H}^2}+\frac{P(1-\lambda^2)}{D}}}\right), \tag{16}$$

with

$$m = a+b-D+\frac{2A_{\rm H}^2}{\Omega_{\rm H}^2} + \frac{P}{D}+2\lambda\sqrt{P/D}[a+1+2\lambda\sqrt{P/D}], \tag{17}$$

$$n = \frac{(a+1)A_{\rm H}^2}{2\Omega_{\rm H}^2}-\lambda\sqrt{DP} - (a+1+2\lambda\sqrt{P/D})[\frac{P}{D}+\frac{A_{\rm H}^2}{2\Omega_{\rm H}^2}]. \tag{18}$$

Applying the adiabatic approximation,[22,23] the transition rates from u_{S1} to u_{S2} and u_{S2} to u_{S1} can be given by[24,25]

$$W_1 = W_{u_{S1}\to u_{S2}} = \frac{\sqrt{|V''(u_{S1})V''(u_{un})|}}{2\pi} \cdot \exp[\frac{\Phi(u_{S1})-\Phi(u_{un})}{D}], \tag{19}$$

$$W_2 = W_{u_{S2}\to u_{S1}} = \frac{\sqrt{|V''(u_{S2})V''(u_{un})|}}{2\pi} \cdot \exp[\frac{\Phi(u_{S2})-\Phi(u_{un})}{D}], \tag{20}$$

with $V'(u)=\frac{dV}{dt}=u^3-(a+1)u^2+(a+b+\frac{3A_{\rm H}^2}{2\Omega_{\rm H}^2}-D)u-\frac{(a+1)A_{\rm H}^2}{2\Omega_{\rm H}^2}+\lambda\sqrt{DP}$, where V'' is the second derivative of V with respect to t. Based on the two-state theory,[20] the discrete random dynamical variable u adopts two possible values u_{S1} and u_{S2}, with probabilities n_1 and n_2 respectively. The master equation governing the evolution of n_1 and n_2 is

$$\frac{dn_1}{dt} = W_2(t)n_2(t)-W_1(t)n_1(t),$$
$$\frac{dn_2}{dt} = W_1(t)n_1(t)-W_2(t)n_2(t). \tag{21}$$

On the assumption that the signal amplitude is small enough, i.e. $A_{\rm L}\ll 1$, the transition rates $W_1(t)$, $W_2(t)$ can be expanded up to the first order of $A_{\rm L}$ as

$$W_1 = \alpha_1-\beta_1 A_{\rm L}\cos(\Omega_{\rm L}t), \quad W_2 = \alpha_2+\beta_2 A_{\rm L}\cos(\Omega_{\rm L}t). \tag{22}$$

The expression of the SNR with respect to the LF signal, defined as the ratio between the power densities of signal to the noise background, can be given by[22,23]

$$SNR = \frac{A_{\rm L}^2\pi(\alpha_1\beta_2+\alpha_2\beta_1)^2}{4\alpha_1\alpha_2(\alpha_1+\alpha_2)}, \tag{23}$$

where

$$\alpha_1 = W_1|_{A_{\rm L}\cos(\Omega_{\rm L}t)=0}, \quad \alpha_2 = W_2|_{A_{\rm L}\cos(\Omega_{\rm L}t)=0}, \tag{24}$$

$$\beta_1 = -\frac{dW_1}{d(A_{\rm L}\cos(\Omega_{\rm L}t))}|_{A_{\rm L}\cos(\Omega_{\rm L}t)=0},$$
$$\beta_2 = -\frac{dW_2}{d(A_{\rm L}\cos(\Omega_{\rm L}t))}|_{A_{\rm L}\cos(\Omega_{\rm L}t)=0}. \tag{25}$$

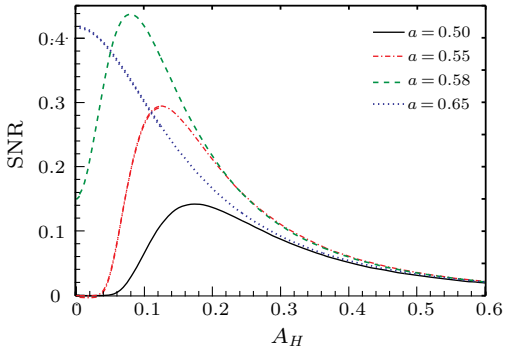

Fig. 1. The SNR versus the amplitude $A_{\rm H}$ of the HF signal for $\lambda=-0.9$, $A_{\rm L}=0.04$, $b=0.23$, $D=0.3$, $P=0.001$, $\Omega_{\rm H}=4$ for different values of the system parameter a.

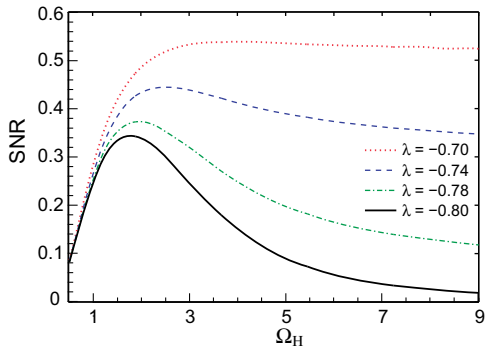

Fig. 2. The SNR versus the frequency $\Omega_{\rm H}$ of the HF signal for $A_{\rm L}=0.1$, $a=0.7$, $b=0.13$, $D=0.5$, $P=0.001$, $A_{\rm H}=0.1$ for different values of the coupling strength λ

SR is a nonlinear phenomenon occurred in a nonlinear system driven by a weak periodic signal and additional noise to the system. When the noise is replaced by an HF periodic force, the system output signal in the response at the LF periodic signal can be maximized for certain value of the amplitude or frequency of the HF signal, thus the VR phenomenon takes place. By virtue of Eq. (23), we investigate the VR and SR phenomena in the FHN model. From Figs. 1–4, we can conclude that the system SNR with respect to the LF signal shows resonance behavior when it is plotted versus the amplitude $A_{\rm H}$ and frequency $\Omega_{\rm H}$, i.e., the VR phenomenon occurs, a similar result was obtained in Ref. [8]. Note that the nonmonotonic behavior of the SNR versus the HF $\Omega_{\rm H}$ is not studied in Ref. [10]. In Fig. 1, the SNR curve is depicted as a function of the amplitude of the HF signal. With the increase of system parameter a, i.e., the

threshold, the peak of the SNR shifts to smaller amplitude $A_{\rm H}$ of the HF signal. This result suggests that for larger threshold a, a relatively smaller HF amplitude should be chosen to ensure the SNR reach its maximum. In addition, the SNR varies non-monotonically with the increase of the threshold a. The peak value of the SNR for $a = 0.58$ is larger than those for $a = 0.55$ and for $a = 0.65$. Moreover, it can be seen from Fig. 2 that the SNR increases monotonically with the growth of the coupling strength λ, thus the coupling strength between the noises can improve the system output signal.

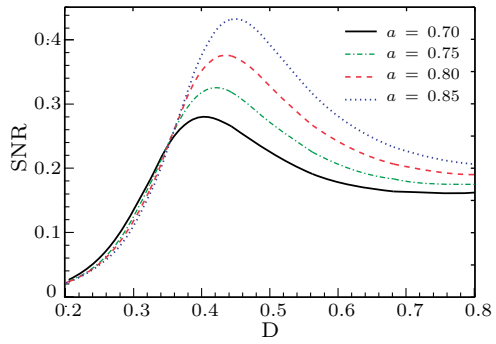

Fig. 3. The SNR versus the multiplicative noise intensity D for $\lambda = 0.01$, $A_L = 0.04$, $b = 0.2$, $P = 0.002$, $A_H = 0.1$, $\Omega_H = 3$ for different values of the system parameter a.

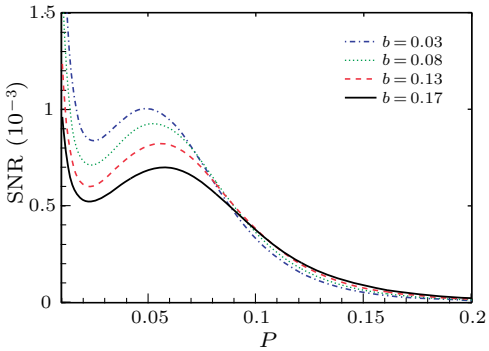

Fig. 4. The SNR versus the additive noise intensity P for $\lambda = -0.5$, $A_{\rm L} = 0.1$, $a = 0.5$, $D = 0.12$, $A_{\rm H} = 0.1$, $\Omega_{\rm H} = 4$ for different values of the system parameter b.

We investigate the influences of the intensities of the multiplicative and additive noises on the SNR from Figs. 3 and 4. It can be seen that the SNR also varies non-monotonically with the growth of the noise intensities, a similar effect was shown in Ref. [19]. The maximum value of SNR shifts to strong noise strength D and P with the increase of the system parameter a and b. This means that for larger system parameters a and b, the SNR will reach its maximum value at a relatively larger amount of noise intensities. The analyzing of Fig. 4 shows that with the growth of the system parameter b, the SNR first reaches a minimum value, after reaching its peak it finally decreases monotonically. Thus the SNR depends non-monotonically on the system parameter b.

In conclusion, we have investigated the VR and SR phenomenon in the FHN model subject to coupled multiplicative and additive noises. Under the adiabatic approximation condition, on the assumption that the frequency of the HF signal is much higher than that of the LF signal, the SNR with regard to the LF signal is obtained. The results show that with the growth of the amplitude and frequency of the HF signal, the SNR exhibits VR behavior. Since HF signal does not appear in the systems of Refs. [12-19], the VR phenomena presented here are new results in the FHN model. The non-monotonous dependence of the SNR on the strengths of the multiplicative and additive noises is also investigated. Furthermore, the SNR presents non-monotonous behavior with the increase of the system parameters, which is not studied in Refs. [12-16]. The monotonous dependence of the SNR on the coupling strength between the multiplicative and additive noise is analyzed.

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