CHIN. PHYS. LETT. Vol. 27, No. 8 (2010) 084703

On the Generalized Continuity Equation

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(Received 9 November 2009)

A generalized continuity equation extending the ordinary continuity equation is found using quanternions to show it is compatible with Dirac, Schrödinger, Klein–Gordon and diffusion equations. This generalized equation is Lorentz invariant. The transport properties of electrons are found to be governed by the Schrödinger-like equation and not by the diffusion equation.

PACS: 47. 10. A−, 47. 75. +f, 47. 90. +a, 47. 10. −g DOI: 10.1088/0256-307X/27/8/084703

The continuity equation governs the conservation of mass/charge/ probability of any closed system. This equation involves the spatial distribution of the flux density that is related to the temporal variation of the particle density (charge/mass). Ordinarily, this equation is derived from the equation of motion. The motion of any continuous charge/mass distribution can be thought of as a continuum (field or fluid). The continuity equation guarantees that there is no loss or gain of such quantities. This equation provides us with information about the system. The information is carried from one point to another by a particle (field) wave. We therefore trust that such an equation should exhibit this property. For electromagnetic field, the information about the motion of the charges is carried away by photons. Because electrons have particle-wave nature, the flow of electrons is not like the flow of classical objects. The information about the movement of electrons is carried by the wave-particle nature that electrons have. The current and charge/mass densities are unified if we formulate our basic partial differential equations governing a particular system in terms of *quaternions*. Thus, when the norm in dealing with the continuity problem is solving partial differential equations, our approach is algebraically-based. Besides producing the known continuity equation, this approach leads to two new equations.

We have recently shown that Maxwell equations can be written as a single quaternionic equation.[1] Moreover, we have found that the quaternionic Lorentz force leads to the Biot–Savart law. The magnetic field produced by the charged particles of the medium is found to be always perpendicular to the particle direction of motion. This magnetic field produces a longitudinal wave called *electroscalar* wave propagates with speed of light in vacuum. Moreover, the electromagnetic field travels with speed of light in vacuum is the presence of a medium if the current and charge densities satisfy the generalized continuity equation.

In this Letter, we investigate a generalized equation expressing a wave-like nature for the charge and current densities, using quaternions and seek a wave-like continuity equation. Such a study will yield the ordinary continuity equation and some other additional equations. This system of equations will reveal the wave nature of the current and charge (mass/probability) densities. We call the set of these equations the generalized continuity equation (GCE). Maxwell, Schrödinger, Klein–Gordon and diffusion equations are shown to be compatible with this GCE. Using the GCE, we have shown that Ohm's law is equivalent to a Schrödinger-like equation. Hence, the electrical properties of metals are consequences of the wave-particle behavior of electrons and not merely due to the drift of electrons.

The flow of any continuous medium is governed by the continuity equation. We would like here to write the continuity equation in terms of quaternions. Quaternions are mathematical construct postulated by Hamilton and were found to have very interesting mathematical properties. They were first used to formulate Maxwell's equations.[2] The multiplication rule for two quaternions, $\widetilde{A} = (a_0, \boldsymbol{A})$ and $\widetilde{B} = (b_0, \boldsymbol{B})$ is given by[3]

$$\widetilde{A}\widetilde{B} = (a_0 b_0 - \boldsymbol{A}\cdot\boldsymbol{B}, a_0\boldsymbol{B} + \boldsymbol{A}b_0 + \boldsymbol{A}\times\boldsymbol{B})\,. \quad (1)$$

Noticed that the quantities a_0, $\boldsymbol{A}$, b_0, $\boldsymbol{B}$ are generally complex functions, a_0 & b_0 are called the scalar parts, and $\boldsymbol{A}$ & $\boldsymbol{B}$ are vector parts of the quaternion. Thus, a single quaternionic equation yields in general four equations. The quaternionic analogue of the covariant ordinary continuity equation ($\partial_\mu J^\mu = 0$) can be written as

$$\widetilde{\nabla}\widetilde{J} = 0. \quad (2)$$

Using Eq. (1), Eq. (2) yields

$$\left[-\left(\boldsymbol{\nabla}\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t}\right), \frac{i}{c}\left(\frac{\partial\boldsymbol{J}}{\partial t} + \boldsymbol{\nabla}\rho\,c^2\right) + \boldsymbol{\nabla}\times\boldsymbol{J}\right] = 0, \quad (3)$$

where

$$\widetilde{\nabla} = \left(\frac{i}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right), \quad \widetilde{J} = (i\rho c, \boldsymbol{J})\,. \quad (4)$$

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Equating real and imaginary parts of the scalar and vector components of Eq. (3) to zero yields

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}+\partial\rho/\partial t=0, \tag{5}$$

$$\boldsymbol{\nabla}\rho+\frac{1}{c^2}\frac{\partial\boldsymbol{J}}{\partial t}=0, \tag{6}$$

$$\boldsymbol{\nabla}\times\boldsymbol{J}=0. \tag{7}$$

We call Eq. (2), or equivalently Eqs. (5)–(7), the generalized continuity equation (GCE). Equation (7) states that the current density $\boldsymbol{J}$ is irrotational.

Following the Jackson four-vector notation,[4] Eqs. (5)–(7) can be written as

$$\partial_\mu J^\mu=0,\quad N_{\mu\nu}\equiv\partial_\mu J_\nu-\partial_\nu J_\mu=0,\quad J_\mu=(\rho c,\boldsymbol{J}). \tag{8}$$

Notice that the tensor $N_{\mu\nu}$ is an antisymmetric tensor. It is evident from Eq. (8) that Eqs. (5)–(7) are invariant under Lorentz transformations. Hence, the GCE is compatible with special theory of relativity. Now differentiating Eq. (5) partially with respect to time and using Eq. (6), we obtain

$$\frac{1}{c^2}\frac{\partial^2\rho}{\partial t^2}-\nabla^2\rho=0. \tag{9}$$

Similarly, taking the divergence of Eq. (6) and using Eq. (5), we obtain

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{J}}{\partial t^2}-\nabla^2\boldsymbol{J}=0, \tag{10}$$

where $\rho=\rho(r,t)$ and $\boldsymbol{J}=\boldsymbol{J}(\boldsymbol{r},t)$. Therefore, both the current density and charge density satisfy the solution of a wave equation propagating with speed of light. Equations (9) and (10) are invariant under Lorentz transformations. We remark that the GCE is applicable to any flow whether created by charged particles or neutral ones.

Maxwell's equations are given by[4]

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=\frac{\rho}{\varepsilon_0},\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0,\quad \boldsymbol{\nabla}\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t}=0,$$
$$\boldsymbol{\nabla}\times\boldsymbol{B}-\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}=\mu_0\boldsymbol{J}. \tag{11}$$

Now differentiating Faraday's equation partially with respect to time and employing Ampere's equation and the vector identity $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{B})-\nabla^2\boldsymbol{B}$ we obtain

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{B}}{\partial t^2}-\nabla^2\boldsymbol{B}=\mu_0\left(\boldsymbol{\nabla}\times\boldsymbol{J}\right). \tag{12}$$

Similarly, differentiating Ampere's equation partially with respect to time and employing Gauss and Faraday equations yields

$$\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}-\nabla^2\boldsymbol{E}=-\frac{1}{\varepsilon_0}\left(\boldsymbol{\nabla}\rho+\frac{1}{c^2}\frac{\partial\boldsymbol{J}}{\partial t}\right). \tag{13}$$

According to the GCE the left hand sides of Eqs. (12) and (13) vanish. This entitles us to say that the electromagnetic field travels with speed of light in free space (when $\rho=0$, $\boldsymbol{J}=0$) and in the presence of charge and current; as far as the current and charge densities satisfy the GCE.

Now consider a conducting media defined by the conduction current density (Ohm's law)

$$\boldsymbol{J}=\sigma\,\boldsymbol{E}. \tag{14}$$

Taking the divergence of Eq. (14) and using the Gauss law, one obtains

$$\boldsymbol{\nabla}\cdot\boldsymbol{J}=\sigma\,\boldsymbol{\nabla}\cdot\boldsymbol{E}=\sigma(\rho/\varepsilon_0). \tag{15}$$

Differentiating Eq. (15) partially with respect to time and using Eq. (6) yield

$$\nabla^2\rho=-\frac{\sigma}{c^2\varepsilon_0}\frac{\partial\rho}{\partial t},\quad \frac{\partial\rho}{\partial t}=-D_c\nabla^2\rho,$$
$$D_c=\frac{c^2\varepsilon_0}{\sigma}=\frac{1}{\mu_0\sigma}. \tag{16}$$

Equation (16) is a Schrödinger-like wave equation describing the motion of free electrons. A metal can be viewed as a gas of free electrons (Fermions). The thermal properties of these electrons are determined by Fermi–Dirac statistics. Thus, Ohm's law satisfies the Schrödinger equation. This is a very interesting result, since the electron motion is governed by Schrödinger equation owing to the wave-particle nature of the electrons. Thus, electrons move in a metal as a material wave (de Broglie wave).

The Drude has proposed a model of electrical conduction of electrons to explain the transport properties of electrons in materials (especially metals).[5] The model, which is an application of kinetic theory, assumes that the microscopic behavior of electrons in a solid may be treated classically and looks much like a pinball machine, with a sea of constantly jittering electrons bouncing and re-bouncing off heavier, relatively immobile positive ions. He has further shown that Ohm's law is true, where he related the current density to the electric field by a linear similar formula. He also connect the current as a response to a time-dependent electric field by a complex conductivity. Moreover, the motion of electrons in metals in not due to diffusion of electrons, the electrons are transported quantum mechanically. Hence, a quantum Ohm's law should be considered instead in which the current density $\boldsymbol{J}$ is obtained from Schrödinger theory of a free electron (Fermi gas). Thus, quantum mechanically, an electron is viewed as a wave traveling through a medium. When the wavelength of the electrons is larger than the crystal spacing, the electrons will propagate freely throughout the metal without collision, therefore their scattering result only from the imperfections in the crystal lattice of the metal.

In Schrödinger terminology the diffusion constant $D_c=\frac{\hbar}{2m\,i}$, which shows that the diffusion constant is complex. This would imply that the conductivity is

complex. In Schrödinger paradigm, the current density and charge density are probabilistic quantities determined by the wave function of the electron (ψ) as[6]

$$\rho = \psi^*\psi, \quad \boldsymbol{J} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*). \tag{17}$$

The current density and the charge here are related by the generalized continuity equation. Therefore, Ohm's current is a probabilistic wave-like current.

It is thought that electrons move in a metal by drifting, but we have seen here that the electrical transport properties of metals are propagated by a material wave rather than drifting of electrons.

Now, we apply Eq. (5) in Eq. (14) and employ Gauss law to obtain

$$\frac{\partial\rho}{\partial t} = -\frac{\sigma}{\varepsilon_0}\rho. \tag{18}$$

Let us write the charge density is a separable form

$$\rho(r,t) \equiv \rho(r)\rho(t). \tag{19}$$

The time dependence of the charge density $\rho(r,t)$ is obtained from Eq. (18) as

$$\rho(t) = \rho_0 \exp\Big(-\frac{\sigma}{\varepsilon_0}\Big)t, \quad \rho_0 = \text{const.} \tag{20}$$

The constant $\tau \equiv \frac{\varepsilon_0}{\sigma}$ is know as the relaxation time, and it is a measure of how fast a conducting medium reaches electrostatic equilibrium. The spatial dependence of the charge density is obtain from substituting Eq. (18) into Eq. (16). This yields the equation

$$\nabla^2\rho(r) - \frac{1}{\lambda_c^2}\rho(r) = 0, \ \lambda_c = \frac{\varepsilon_0 c}{\sigma} = \frac{1}{\mu_0 c\sigma} = \frac{D_c}{c}. \tag{21}$$

The solution of Eq. (21) is given by

$$\rho(r) = A\exp\Big(-\frac{r}{\lambda_c}\Big), \quad A = \text{const.} \tag{22}$$

Therefore, the charge density distribution (spacetime) is given by

$$\rho(r,t) = Be^{-\frac{1}{\lambda_c}(r+ct)} + Fe^{-\frac{1}{\lambda_c}(-r+ct)}, \ B,F = \text{const.} \tag{23}$$

This shows that the charges density decays in space and time. However, if ω_c is complex, we will have an oscillatory charge density solution. This is true if we consider σ to be complex. The spreading of wavepackets in quantum mechanics is directly related to the spreading of probability densities in diffusion.

Now taking the gradient of Eq. (15) and employing Eq. (6) and the vector identity $\nabla\times(\boldsymbol{\nabla}\times\boldsymbol{J}) = \nabla(\boldsymbol{\nabla}\cdot\boldsymbol{J}) - \nabla^2\boldsymbol{J}$ [7] we obtain

$$\partial\boldsymbol{J}/\partial t = -D_c\nabla^2\boldsymbol{J}. \tag{24}$$

This equation shows that $\boldsymbol{J}$ is governed by the Schrödinger-like equation. Hence, both the current density and charged density are governed by a Schrödinger-like equation.

Now applying Eq. (14) in Eq. (6) and using Eq. (11) yield

$$\partial\boldsymbol{E}/\partial t = -D_c\nabla^2\boldsymbol{E}, \tag{25}$$

where we have used the vector identity $\nabla\times(\boldsymbol{\nabla}\times\boldsymbol{E}) = \nabla(\boldsymbol{\nabla}\cdot\boldsymbol{E}) - \nabla^2\boldsymbol{E}$ and the fact that $\boldsymbol{\nabla}\times\boldsymbol{E} = 0$. This equation can also be obtained directly from Eq. (24) by dividing both sides of Eq. (24) by σ and using Eq. (14).

Equation (21) may define a limiting conductivity of a material (σ_0), when we equate λ_c to Compton wavelength of the electron of mass m_e. In this case one finds

$$\sigma_0 = m_e/(\mu_0 h). \tag{26}$$

This amounts to a value of $\sigma_0 = 1.09\times10^9\,\Omega^{-1}\,\text{m}^{-1}$. This can be compared with the best conductivity of metal (Silver) which is $6.3\times10^7\,\Omega^{-1}\,\text{m}^{-1}$. Thus, the maximum possible conductivity is set by quantum mechanics and governed by Eq. (26). Because of this, the conductivity at zero Kelvin is never infinite but should be limited to σ_0.

Diffusion is a transport phenomena resulting from random molecular motion of molecules from a region of higher concentration to a lower concentration. The result of diffusion is a gradual mixing of material. Diffusion is of fundamental importance in physics, chemistry, and biology.

The diffusion equation is given by[9]

$$\boldsymbol{J} = -D\boldsymbol{\nabla}\rho, \tag{27}$$

where D is the diffusion constant. This is known as Fick's law. Taking the divergence of Eq. (27) and using Eq. (5), one finds

$$\partial\rho/\partial t = D\nabla^2\rho. \tag{28}$$

This shows that the density ρ satisfies the Diffusion equation. The normalized solution to Eq. (28) is

$$\rho(x,t) = \frac{1}{\sqrt{4\pi D\,t}}\exp\Big(-\frac{x^2}{4Dt}\Big), \tag{29}$$

which when applied in Eq. (27) yields the current density $\boldsymbol{J}$. It is obvious that the current in Eq. (27) satisfies the GCE, viz., Eq. (5)–(7). Differentiation Eq. (27) partially with respect to time and employing Eqs. (5)–(7) and the vector identity, $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{J}) = \nabla^2\boldsymbol{J} + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{J})$ [6] one obtains

$$\partial\boldsymbol{J}/\partial t = D\nabla^2\boldsymbol{J}. \tag{30}$$

Hence, both ρ and $\boldsymbol{J}$ satisfy the diffusion equation.

We would like here to show that the generalized continuity equations, viz., Eq. (8) are compatible with Dirac equation. To this aim, we apply the current density four-vector according to Dirac formalism, i.e., $J_\mu = \bar\psi\gamma_\mu\psi$ in Eq. (8). This yields[5]

$$\partial^\mu J^\nu - \partial^\nu J^\mu = (\partial^\mu\bar\psi)\gamma^\nu\psi + \bar\psi\gamma^\nu\partial^\mu\psi - (\partial^\nu\bar\psi)\gamma^\mu\psi - \bar\psi(\gamma^\mu\partial^\nu\psi) = 0. \tag{31}$$

The first term in the above equation can be written as

$$(\partial^\mu\bar{\psi})\gamma^\nu\psi = (\partial^\mu\psi^+)\gamma^0\gamma^\nu\psi$$
$$= (\partial^\mu\psi^+)\gamma^{\nu+}\gamma^0\psi = (\gamma^\nu\partial^\mu\psi)^+\gamma^0\psi. \quad (32)$$

Hence,

$$(\partial^\mu\bar{\psi})\gamma^\nu\psi = (\bar{\psi}\gamma^\nu\partial^\mu\psi)^+. \quad (33)$$

Similarly,

$$(\partial^\nu\bar{\psi})\gamma^\mu\psi = (\bar{\psi}\gamma^\mu\partial^\nu\psi)^+. \quad (34)$$

Substituting these terms in Eq. (31) completes the proof, since for a plane wave $\partial^\mu\psi = ik^\mu\psi$.

The Klein–Gordon equation describes the motion of scalar particles with integral spin. For such a system the current density four-vector is given by[5]

$$J_\mu = \frac{\hbar}{2mi}\left(\phi^*\partial_\mu\phi - (\partial_\mu\phi^*)\phi\right), \quad (35)$$

so that the generalized continuity equation becomes

$$\partial_\mu J_\nu - \partial_\nu J_\mu = \frac{\hbar}{2mi}\left[\partial_\mu(\phi^*\partial_\nu\phi - (\partial_\nu\phi^*)\phi) - \partial_\nu(\phi^*\partial_\mu\phi - (\partial_\mu\phi^*)\phi)\right] = 0, \quad (36)$$

and hence, Klein–Gordon equations is compatible with our GCE too.

It is the Schrödinger equation governing the motion of non-relativistic particles, e.g. electrons. The current density and probability density in Schrödinger formalism are given by[5]

$$\rho = \psi^*\psi, \quad \boldsymbol{J} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - (\nabla\psi^*)\psi). \quad (37)$$

Applying Eq. (6), one obtains

$$\boldsymbol{\nabla}\times\boldsymbol{J} = \frac{\hbar}{2\,mi}\left[(\boldsymbol{\nabla}\times(\psi^*\nabla\psi) - \boldsymbol{\nabla}\times(\nabla\psi^*)\psi)\right]. \quad (38)$$

Using the two vector identities $\boldsymbol{\nabla}\times(f\boldsymbol{A}) = f(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\times(\boldsymbol{\nabla}f)$ and $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f) = 0$, Eq. (38) vanishes, and hence one of the GCE is satisfied. We now apply Eq. (37) in Eq. (6) to obtain

$$\boldsymbol{\nabla}(\rho c^2) + \frac{\partial\boldsymbol{J}}{\partial t} = \boldsymbol{\nabla}(\psi^*\psi) + \frac{\hbar}{2\,mi}\frac{\partial}{\partial t}\left[\psi^*\nabla\psi - (\nabla\psi^*)\psi\right] = 0. \quad (39)$$

Upon using the Schrödinger equation,

$$H\psi = i\hbar\frac{\partial\psi}{\partial t}, \qquad \psi^*H = -i\hbar\frac{\partial\psi^*}{\partial t}. \quad (40)$$

Equation (39) vanishes for a plane wave equation, i.e., $\psi(r,t) = A\exp i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$, $A = \text{const}$. Therefore, the GCE is satisfied by Dirac, Klein–Gordon and Schrödinger equations. This implies that these GCE is indeed fundamental in formulating any field theoretic models involving the motion of particles or fluids. This equation will have immense consequences when applied to the theory of electrodynamics. Such an application will lead to better formulation and understanding of the astrophysical phenomena pertaining to the evolution of dense objects.

We have derived the generalized continuity equations and shown how it is applicable to the particle flow. We have shown that the generalized continuity equation is lorentz invariant. We have then shown that the basic equations of motion are compatible with the generalized continuity equation. The diffusion equation is in agreement with the GCE. Moreover, the current and density in Dirac, Klein–Gordon and Schrödinger formalisms are compatible with the GCE too. The classical Ohm's law is found to be compatible with the Schrödinger-like equation. Transport properties of metals are shown to be propagated by a material wave rather than drifting and diffusion.

One of us (AIA) would like to thank the University of Khartoum for financial support to this work and the Sultan Qaboos university for hospitality where this work is carried out. We would also like to thank the anonymous referees for their critical and useful comments.

References

[1] Arbab A I and Satti Z A 2009 *Prog. Phys.* **8** 2
[2] Hamilton W R 1844 *Philosophical Magazine* **25** 489
[3] Tait P G 1873 *An Elementary Treatise on Quaternions* 2nd edn (Cambridge: Cambridge University)
Kelland P and Tait P G 1904 *Introduction to Quaternions* 3rd edn (London: Macmillan)
[4] Jackson J D 1975 *Classical Electrodynamics* 2nd edn (New York: John Wiley)
[5] Drude P 1900 *Annalen der Physik* **306** 566
[6] Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw-Hill)
[7] Lawden D F 1968 *Tensor Calculus and Relativity* (London: Methuen)
[8] Arbab A I 2009 *14th International Conference on Modelling Fluid Flow (CMFF'09)* (Budapest, Hungary 9–12 September 2009)
[9] Fick A 1855 *Ann. Phys. (Leipzig)* **170** 59