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# Dynamics of Fokker–Planck Equation with Logarithmic Coefficients and Its Application in Econophysics

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*We demonstrate that Fokker–Planck equations with logarithmic factors in diffusion and drift terms can be straightforwardly derived from the class of "constant elasticity of variance" stochastic processes without appealing to any symmetry argument. Analytical closed-form solutions are available for some special cases of this class of Fokker–Planck equations. The dynamics of the underlying stochastic variables are examined. These Fokker–Planck equations have found a rather wide range of applications in various contexts. In particular, in the field of econophysics we have demonstrated their immediate relevance to modelling the exchange rate dynamics in a target zone, e.g. the linked exchange rate system of the Hong Kong dollar. Furthermore, the knowledge of exact solutions in some special cases can be useful as a benchmark to test approximate numerical or analytical procedures.*

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The one-dimensional Fokker–Planck equation (FPE) with variable coefficients

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}[B(x,t)P(x,t)] - \frac{\partial}{\partial x}[A(x,t)P(x,t)], \tag{1}$$

provides a very useful tool for modelling a wide variety of stochastic phenomena arising in many areas of physics, chemistry, biology, engineering and finance.[1−3] As is well known, while the stationary solution of the FPE can be given in closed form (at least up to quadratures) if the condition of detailed balance holds, the study of its time-dependent solution is a much more complicated problem. Since there are only a limited number of cases that can be solved analytically, this situation gives rise to many stimulating opportunities for the development of approximate methods to analyze such problems. Accordingly, much research work has been carried out to better understand the FPE with non-trivial drift and diffusion terms.

Recently, based upon the symmetry properties of the FPEs, Pesz[4] derived a class of FPEs with logarithmic factors in diffusion and drift terms. Exact algebraic and log-algebraic solutions were also obtained. Such logarithmic dependence has found applications in various contexts, e.g. the mobility functions for diffusing hard spheres,[5] quantum chaos diffusion processes,[6] systems with spatial confinement like a Brownian walker trapped between fixed plates[7] and tumour cell growth with a bounded size of tumour cells.[8] Further generalizations by means of symmetry arguments were subsequently reported by Silva *et al.*[9] and Cardeal *et al.*,[10] too. In this Letter we demonstrate that, without appealing to symmetry arguments, the class of FPEs with logarithmic factors in diffusion and drift terms can be straightforwardly derived from the so-called "square-root" stochastic process.[11] In a similar manner we are also able to extend the approach to obtain FPEs with more general logarithmic dependence from the class of "constant elasticity of variance" stochastic processes,[11] of which the square-root process is a special case.

Moreover, in view of the growing interest in the applications of physical methods to research in economics (as evidenced by the formation of econophysics) recently,[3] we try to apply these results to construct a new simple and analytically tractable model of the exchange rate dynamics in a target zone, in which the central bank involved has promised to keep the exchange rate within a zone,[12−14] e.g. the linked exchange rate system of the Hong Kong dollar. At the strong-side limit of the target zone where the exchange rate is defined as a domestic currency value of a unit of foreign currency, the central bank can always keep the exchange rate within the zone by selling the domestic currency to maintain the exchange rate at or higher than the strong-side limit. At the weak-side limit the central bank however needs to buy the domestic currency and thus consumes its foreign exchange reserves to prevent the exchange rate from breaching the weak-side limit. This means that the exchange rate could be bounded within the zone provided that the central bank has adequate foreign exchange reserves. As long as the exchange rate is "well within" the band, market participants behave as if they are in a comfort zone and do not feel particularly compelled or encouraged to pull the exchange rate towards the central parity. However, when the exchange rate moves closer to the boundary, the market may anticipate an intervention and stop the rate or even push the rate back. Hence, we have a bounded process equipped with a stopping/reversion effect that only occurs close to the boundaries of the zone.

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In this Letter, first we show explicitly how the class of FPEs with logarithmic factors in diffusion and drift terms can be derived from the class of the constant elasticity of variance stochastic processes. Then, we apply these results to model the exchange rate dynamics in a target zone. Finally, a brief summary of this study is presented.

We consider the square-root stochastic process defined by the stochastic differential equation (SDE):[11]

$$dS = \kappa(\theta - S)dt + \sigma\sqrt{S}dZ, \tag{2}$$

for $S \in [0, \infty)$, where $\sigma^2 S$ is the variance, $dZ$ is a standard Wiener process, and $\kappa$ determines the speed of the mean-reverting drift towards the long-term mean $\theta$. The conditional probability density function (PDF) $P(S, t; S_0, t_0)$ is governed by the FPE:

$$\frac{\partial P(S,t;S_0,t_0)}{\partial t} = \frac{\partial^2}{\partial S^2}\left[\left(\frac{1}{2}\sigma^2 S\right)P(S,t;S_0,t_0)\right] - \frac{\partial}{\partial S}[\kappa(\theta - S)P(S,t;S_0,t_0)]. \tag{3}$$

Introducing a change of variables, namely $S = -\ln x$ and $S_0 = -\ln x_0$, the FPE in Eq. (3) can be rewritten as

$$\begin{aligned}&\partial p(x,t;x_0,t_0)/\partial t \\ &= \frac{\partial^2}{\partial x^2}\left[\left(\frac{1}{2}\sigma^2 x^2(-\ln x)\right)p(x,t;x_0,t_0)\right] \\ &\quad - \frac{\partial}{\partial x}\left\{\left[\kappa(\ln x_\theta - \ln x)x + \frac{1}{2}\sigma^2(-\ln x)x\right]\right. \\ &\quad \left.\times\, p(x,t;x_0,t_0)\right\}\end{aligned} \tag{4}$$

for $x \in [0, 1]$, where $x_\theta = \exp(-\theta)$ and $p(x, t; x_0, t_0) = P(S, t; S_0, t_0)/x$. Thus, we have demonstrated that Pesz's class of FPEs with logarithmic factors in diffusion and drift terms can be derived from the square-root process. The conditional PDF $P(S, t; S_0, t_0)$ is known to be[11]

$$\begin{aligned}&P(S,t;S_0,t_0) \\ &= \frac{2}{\sigma^2 C_1\,(t-t_0)}\left(\frac{S}{S_0}\right)^{\omega/2}\exp\left\{-\frac{\omega+2}{2}C_2(t-t_0)\right\} \\ &\quad\times\exp\left(-\frac{2S_0 + 2S\exp\{-C_2(t-t_0)\}}{\sigma^2 C_1(t-t')}\right) \\ &\quad\times I_\omega\left(\frac{4S_0^{1/2}S^{1/2}\exp\left\{-\frac{1}{2}C_2(t-t_0)\right\}}{\sigma^2 C_1(t-t_0)}\right),\end{aligned} \tag{5}$$

where

$$\omega = \frac{2\kappa\theta}{\sigma^2} - 1, \quad C_2(\tau) = -\kappa\tau, \quad C_1(\tau) = \frac{\exp(\kappa\tau) - 1}{\kappa}, \tag{6}$$

and $I_\omega$ is the modified Bessel function of the first kind of order $\omega$. Since the diffusion coefficient $\sigma^2 x^2(-\ln x)/2$ vanishes at both boundaries, the stochastic variable $x$ is expected to be bounded within the interval $[0, 1]$. It is, in fact, not difficult to show that provided $\sigma^2 < 4\kappa\theta$, there is no probability leakage and the total probability is preserved. Otherwise, there exists a noise-induced probability current through the boundary at the origin. Hence, it is a quasi-bounded process.

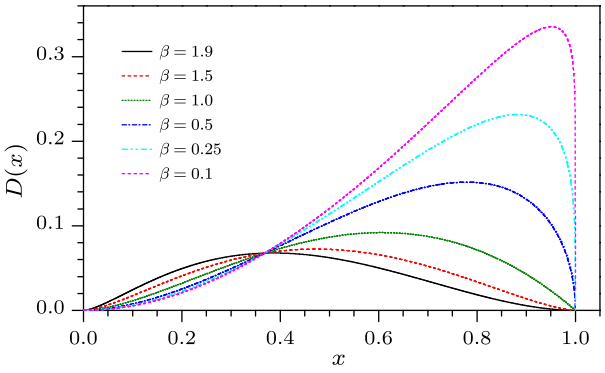



**Fig. 1.** The diffusion coefficient $D(x) \equiv \sigma^2(-\ln x)^\beta x^2/2$ versus $x$ for different values of $\beta$ with $\sigma = 1$.

Next, we would like to investigate if FPEs with more general logarithmic dependence can be generated similarly. In order to achieve this, we consider the class of constant elasticity of variance stochastic processes:[11]

$$dS = \kappa(\theta - S)dt + \sigma S^{\beta/2}dZ, \tag{7}$$

for $S \in [0, \infty)$ and $0 < \beta < 2$, of which the square-root process is a special case. The conditional PDF $P(S, t; S_0, t_0)$ satisfies the FPE:

$$\begin{aligned}&\partial P(S,t;S_0,t_0)/\partial t \\ &= \frac{\partial^2}{\partial S^2}\left[\left(\frac{1}{2}\sigma^2 S^\beta\right)P\,(S,t;S_0,t_0)\right] \\ &\quad - \frac{\partial}{\partial S}\left[\kappa(\theta - S)P(S,t;S_0,t_0)\right].\end{aligned} \tag{8}$$

In terms of the stochastic variable $x$ and distribution function $p(x, t; x_0, t_0)$ defined above, we can re-express this FPE as

$$\begin{aligned}&\partial p(x,t;x_0,t_0)/\partial t \\ &= \frac{\partial^2}{\partial x^2}\left[\left(\frac{1}{2}\sigma^2 x^2(-\ln x)^\beta\right)p(x,t;x_0,t_0)\right] \\ &\quad - \frac{\partial}{\partial x}\left\{\left[\kappa(\ln x_\theta - \ln x)x + \frac{1}{2}\sigma^2(-\ln x)^\beta x\right]\right. \\ &\quad \left.\times\, p(x,t;x_0,t_0)\right\}.\end{aligned} \tag{9}$$

Consequently, we have succeeded in deriving a class of FPEs with more general logarithmic factors in diffusion and drift terms from the class of constant elasticity of variance processes.

As shown in Fig. 1, the diffusion coefficient $\sigma^2(-\ln x)^\beta x/2$ reaches its maximum at $x = \exp(-\beta/2)$ and vanishes at both boundaries, implying

that the stochastic process could be bounded. Unfortunately, for the general case of $\beta \neq 1$ and $\theta \neq 0$, the solution of Eq. (9) is not available in an analytical form. In order to facilitate a better understanding of the behaviour of the stochastic process of $x$, we introduce the stochastic variable $\xi = S^{(2-\beta)/2}$, which can be easily shown via Ito's lemma to obey the SDE:[2]

$$d\xi = \left(\frac{2-\beta}{2}\right)\left\{\kappa\theta\xi^{-\beta/(2-\beta)} - \kappa\xi - \frac{1}{4}\beta\sigma^2\xi^{-1}\right\}dt + \left(\frac{2-\beta}{2}\right)\sigma dZ, \tag{10}$$

for $\xi \in [0, \infty)$. It is not difficult to see that for $0 < \beta < 1$ there exists an absorbing boundary at the origin, whereas for $1 < \beta < 2$ the term containing the long-term mean $\theta$ becomes the dominant term of the drift coefficient and forms an infinite barrier against the stochastic variable $\xi$ breaching the boundary at the origin. Moreover, the drift coefficient approaches $-\infty$ asymptotically as $\xi \to \infty$ for $0 < \beta < 2$, indicating that there is a non-attractive natural boundary at infinity. Hence, the special case of $\beta = 1$, i.e. the square-root process, serves as the watershed separating these two different regimes, namely a bounded process for $1 < \beta < 2$ and an unbounded process for $0 < \beta < 1$.

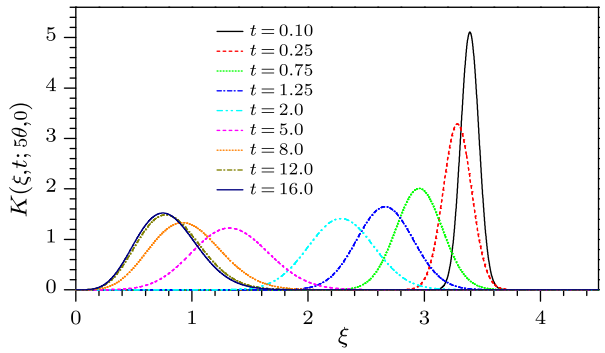



**Fig. 2.** The PDF $K(\xi, t; 5\theta, 0)$ versus $\xi$ at time $t = 0.1$, 0.25, 0.75, 1.25, 2, 5, 8, 12, 16. Other input parameters are $\theta = -\ln(0.5)$, $\sigma = 0.5$ and $\kappa = 0.45$.

For illustration, in Fig. 2 we plot the time evolution of the PDF $K(\xi, t; \xi', t')$:

$$\begin{aligned} K(\xi,t;\xi',t') &= 2\xi P(S,t;S_0,t_0) \\ &= \frac{4\xi}{\sigma^2 C_1(t-t')}\left(\frac{\xi}{\xi'}\right)^{\omega}\exp\left\{-\frac{\omega+2}{2}C_2(t-t')\right\} \\ &\quad\times\exp\left(-\frac{2\xi'^2+2\xi^2\exp\{-C_2(t-t')\}}{\sigma^2 C_1(t-t')}\right) \\ &\quad\times I_\omega\left(\frac{4\xi'\xi\exp\{-\frac{1}{2}C_2(t-t')\}}{\sigma^2 C_1(t-t')}\right) \end{aligned} \tag{11}$$

for a typical case of the quasi-bounded square-root process, in which $\theta = -\ln(0.5)$, $\xi' = 5\theta$, $t' = 0$, $\sigma = 0.5$ and $\kappa = 0.45$. It is observed that the bell-shaped PDF drifts from the initial position towards the long-term mean steadily and that the noise-induced spreading of the PDF is fairly moderate. Since no probability leakage exists at the origin in this case, the probability that the stochastic variable $\xi$ is found close to the long-term mean tends to accumulate gradually, and the PDF will eventually approach the steady-state limit:

$$K(\xi, t\to\infty; \xi', t') = \frac{2\xi^{2\omega+1}}{\Gamma(\omega+1)}\left(\frac{\kappa}{\sigma^2}\right)^{\omega+1}\exp\left\{-\frac{\kappa}{\sigma^2}\xi^2\right\}, \tag{12}$$

where $\Gamma$ is the Gamma function. It is also easy to realize that this steady-state PDF corresponds to the well-known Gamma distribution in the stochastic variable $\xi^2$ and has a peak located at $\xi = \sqrt{\theta - \sigma^2/(4\kappa)}$.[2]

Finally, it should be noted that for $\beta \neq 1$ and $\theta = 0$, the SDE resembles the one of the square-root process, *i.e.* the special case of $\beta = 1$ and $\theta \neq 0$, and that the corresponding PDF can be readily found to be

$$\begin{aligned} &K(\xi,t;\xi',t') \\ &= \frac{4\xi}{(2-\beta)^2\sigma^2 C_1(t-t')}\left(\frac{\xi}{\xi'}\right)^{\omega}\exp\left\{-\frac{\omega+2}{2}C_2(t-t')\right\} \\ &\quad\times\exp\left(-\frac{2\xi'^2+2\xi^2\exp\{-C_2(t-t')\}}{(2-\beta)^2\sigma^2 C_1(t-t')}\right) \\ &\quad\times I_\omega\left(\frac{4\xi'\xi\exp\{-\frac{1}{2}C_2(t-t')\}}{(2-\beta)^2\sigma^2 C_1(t-t')}\right), \end{aligned} \tag{13}$$

where

$$\omega = -\frac{1}{2-\beta}, \quad C_2(\tau) = -(2-\beta)\kappa\tau, \quad C_1(\tau) = \frac{\exp\{(2-\beta)\kappa\tau\}-1}{(2-\beta)\kappa}. \tag{14}$$

It is obvious that the boundary at the origin is an absorbing boundary in these cases. In other words, if the long-term mean $\theta$ is fixed at the origin, then the stochastic process is no longer bounded and probability leakage occurs at the origin.

We now demonstrate how to apply the present results to model the exchange rate dynamics in a target zone.[12–14] The exchange rate $\mathcal{S}$ (defined as a domestic currency value of a unit of foreign currency) of a target zone which, in the absence of realignment risk, is supposed to follow a random walk most of the time, and tend to bounce off the strong-side and weak-side limits (i.e. $\mathcal{S}_L$ and $\mathcal{S}_U$, respectively) whenever it comes close to them. With no loss of generality, we can define a normalized dimensionless exchange rate $R \in [0, 1]$ as follows:

$$R \equiv \frac{\mathcal{S}-\mathcal{S}_L}{\mathcal{S}_U-\mathcal{S}_L}, \tag{15}$$

which essentially provides a measure of the distance of the exchange rate $\mathcal{S}$ from the strong-side limit $\mathcal{S}_L$. Then $X \equiv -\ln R$ must satisfy that $0 \leq X < \infty$ with the zone boundaries $\mathcal{S}_U$ and $\mathcal{S}_L$ being mapped to $X = 0$ and $X = \infty$, respectively. As a model for the logarithm $X$, we propose that $X$ follows the constant

elasticity of variance (CEV) diffusion process defined by[11]

$$dX = \kappa(\theta - X)dt + \sigma X^{\beta/2}dZ, \quad 0 < \beta < 2 \qquad (16)$$

where $\sigma^2 X^\beta$ is the variance that depends upon the level of $X$, $dZ$ is a standard Wiener process, and $\kappa$ determines the speed of the mean-reverting drift towards the long-term mean $\theta$. When the process approaches the boundary at $X = 0$ (i.e. the weak-side limit), the variance becomes small and the mean-reverting drift drives the process away from the boundary. On the other hand, as the instantaneous variance $\sigma^2 X^{\beta-2}$ of the fractional change in $X$ (i.e. $dX/X$) is a decreasing function of $X$, the boundary at $X = \infty$ (i.e. the strong-side limit) is inaccessible, especially in the presence of the mean-reverting drift.

In order to facilitate a better understanding of the dynamics of the stochastic process of $X$, we now take a closer look at the normalized dimensionless exchange rate $R$ which can be easily shown via the Ito's lemma to obey the SDE:

$$dR = \left\{\kappa(\ln R_\theta - \ln R) + \frac{1}{2}\sigma^2(-\ln R)^\beta\right\} Rdt + \sigma R(-\ln R)^{\beta/2}dZ, \qquad (17)$$

where $R_\theta \equiv \exp(-\theta)$. The variance $\sigma^2 R^2(-\ln R)^\beta$ achieves its maximum at $R = \exp(-\beta/2)$ and vanishes at the zone boundaries, implying that the stochastic process is bounded. The stochastic process of $R$ is manifestly asymmetrical about the central parity, i.e. $R = 0.5$, reflecting the different nature of the strong-side and weak-side limits and the corresponding intervention measures. It is also observed that for $\beta < 1$ the stochastic process is much more volatile near the boundary at $R = 1$, signalling the possibility of breaching the weak-side limit. As a result, the target zone may no longer be bounded, and the boundary at $R = 1$, or equivalently $X = 0$, may now become penetrable.

The proposed exchange rate dynamics allows different boundary conditions at the weak-side and strong-side limits because a central bank's actions and capability of keeping the exchange rate within the zone at these two limits are very different. The intervention policy of the central bank and the behaviour of market participants are modelled implicitly by specifying the drift and diffusion coefficients of the exchange rate dynamics that determine the boundary conditions at the two limits. There is an explicit inaccessible boundary at the strong-side limit where the intervention policy always functions by simply selling the domestic currency to the market. The variance of the exchange rate also vanishes at the strong-side limit. Such a property is similar to the bounded exchange rate dynamics in Ref. [15] in which the variance of the exchange rate vanishes at both the weak-side and strong-side limits under the assumption that the central bank will intercede to stabilise the foreign exchange market and keep the exchange rate well within the zone.

Nevertheless, the proposed dynamics are different from the model of Ref. [15] that while the variance of the exchange rate declines towards the weak-side limit, the exchange rate could breach the limit under particular conditions, i.e. realignment could occur and exchange rate is thus quasi-bounded at the weak-side limit. The possible realignment conditions are determined by the values of the drift and diffusion coefficients of the exchange rate dynamics, which can incorporate market data such as interest rates and implied volatility from the option market. This specification is consistent with empirical evidence in Ref. [16] that most realignments occur when the exchange rate is close to the weak-side limit of the target zone. The exchange rate dynamics are therefore quasi-bounded and different from the Krugman-type models that the associated realignment risk takes place only at the weak-side limit.

In summary, we have demonstrated that FPEs with logarithmic factors in diffusion coefficient and drift coefficient can be straightforwardly derived from the class of constant elasticity of variance stochastic processes without appealing to any symmetry argument. Analytical closed-form solutions are available for some special cases of this class of FPEs. The dynamics of the underlying stochastic variables are examined. Further generalizations can also be easily obtained by substituting the drift coefficient in Eq. (7) by a more general function of the underlying stochastic variable. These FPEs are believed to have a rather wide range of applications. For instance, we have demonstrated that the analysis can be applied to model the exchange rate dynamics in a target zone, e.g. the linked exchange rate system of the Hong Kong dollar. Furthermore, the knowledge of analytical solutions in some special cases can be useful as a benchmark to test approximate numerical or analytical procedures.

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