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Absorption Cross Section and Decay Rate of Stationary Axisymmetric Einstein–Maxwell Dilaton Axion Black Hole *

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The low energy absorption cross section and the decay rate of the stationary Axisymmetric Einstein–Maxwell Dilaton Axion black hole for massless scalar particles is calculated analytically. It is shown that the partial absorption cross section increases as the rotating parameter a and the absolute value of the dilaton D decreases. It is also shown that the partial absorption cross section is not always positive due to superradiance factor $\omega - m\Omega$. However, the decay rate of this black hole is always positive.

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It is well known that scattering is a powerful tool conceptually as well as experimentally in quantum mechanics. This tool is also crucial to the understanding the properties of the curved spacetime. Especially, on the black hole background, a considerable effort has gone into studies of waves scattered off black holes. Both numerical and analytical methods in solving the various wave equations[1,2] in black hole scattering have been developed.[3]

A feature of obvious importance in black hole scattering is absorption. The absorption cross section in four-dimensional black holes has been studied extensively since the 1970s. The first attempt to calculate analytical absorption coefficients and cross section was made by Starobinsky[4] and many subsequent authors (such as Gibbons[5] Page[7] and Unruh[6]) in the low energy limit. In these calculations if we consider the particle to be a massless minimally coupled scalar, then we can find that the cross section equals the area of the horizon of the black hole in the zero-frequency limit. In a series of papers, Sanchez[8] considered the scattering and absorption of massless scalar particles by a Schwarzschild black hole with numerical methods. It can be shown that the total absorption cross section (as a function of incident frequency) exhibits oscillations around the geometric-optics limit characteristic of diffraction patterns. recently, the computation of the low-energy absorption cross section has become an interesting subject in the context of string theories and high dimensional spacetime. Thus it is natural to ask whether the low-energy absorption cross sections for massless particles holds for the same result in high-dimensional spacetime. In fact, Das *et al.* have shown[9] that this equality is valid for arbitrary dimensional spherically symmetric black hole, which is an universal property for minimally coupled massless particles. This universality are examined and generalized in the arbitrary-dimensional black hole and higher-dimensional objects, such as p-brane.[10−13]

Previously, we investigated the absorption cross section of the static Einstein–Maxwell Dilaton Axion (EMDA) black hole for scalar and dirac particles and found some interesting new properties.[14] In this Letter, we calculate the low energy absorption cross section and the decay rate of the Stationary Axisymmetric EMDA black hole[15] for massless scalar particles by using the coefficients matching method.

The metric of the stationary Axisymmetric EMDA black hole[16] in the Boyer–Lindquist coordinates can be expressed as

$$\begin{aligned} ds^2 = &-\frac{\Sigma - a^2\sin^2\theta}{\Delta}dt^2 - \frac{2a\sin^2\theta}{\Delta}(\Xi - \Sigma)\,dtd\varphi \\ &+ \frac{\Delta}{\Sigma}dr^2 + \Delta d\theta^2 \\ &+ \frac{\sin^2\theta}{\Delta}\left(\Xi^2 - \Sigma a^2\sin^2\theta\right)d\varphi^2, \end{aligned} \tag{1}$$

with

$$\begin{aligned} &\Sigma = r^2 - 2Mr + a^2 = (r - r^-)(r - r^+), \\ &\Delta = r^2 - 2Dr + a^2\cos^2\theta, \quad \Xi = r^2 - 2Dr + a^2, \\ &r_\pm = M_{\rm ADM} + D \pm \sqrt{(M_{ADM} + D)^2 - a^2}, \end{aligned} \tag{2}$$

where D, M and a represent the dilaton, mass and the angular momentum per unit mass of the stationary axisymmetric EMDA black hole, respectively. $M_{\rm ADM} = M - D$ is the ADM mass, r_- and r_+ are the inner and outer horizon of the black hole. Throughout this study we use $G = c = 1$. The area A_H, Hawking temperature T_H and angular velocity Ω at the horizon are found to be[16]

$$\begin{aligned} A_H &= 4\pi(r_+^2 - 2Dr_+ + a^2), \\ T_H &= \frac{r_+^2 - a^2}{4\pi r_+(r_+^2 - 2Dr_+ + a^2)}, \\ \Omega &= \frac{a}{r_+^2 - 2Dr_+ + a^2}. \end{aligned} \tag{3}$$

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The Klein–Gordon equation in a curved space time is

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial\chi^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\phi}{\partial\chi^{\nu}}\right)\phi=0. \tag{4}$$

Thus we obtain a master wave equation of the stationary axisymmetric EMDA spacetime

$$\begin{aligned}&\frac{\Xi^2-\Sigma a^2\sin^2\theta}{\Sigma}\frac{\partial^2\Phi}{\partial t^2}-\frac{\partial}{\partial r}\left(\Sigma\frac{\partial\Phi}{\partial r}\right)\\&-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)-\frac{\Sigma-a^2\sin^2\theta}{\sin^2\theta\Sigma}\frac{\partial^2\Phi}{\partial\varphi^2}\\&+\frac{4a(M-D)r}{\Sigma}\frac{\partial^2\Phi}{\partial t\partial\varphi}\phi=0.\end{aligned} \tag{5}$$

If we let $\phi=R(r)S(\theta)e^{im\phi-i\omega t}$, Eq. (5) can be separated. Thus, we can obtain the radial wave equation $R(r)$

$$\begin{aligned}&\Sigma\frac{d}{dr}\left(\Sigma\frac{dR}{dr}\right)+\left[\Xi^2\omega^2-4a\omega m(M-D)r\right.\\&\left.+m^2a^2\right]R(r)-\left[\omega^2a^2+\lambda_{lm}\right]\Sigma R(r)=0,\end{aligned} \tag{6}$$

and the angular equation $S(\theta)$

$$\begin{aligned}&\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{S}{d\theta}\right)+\left(\lambda_{lm}+(a\omega)^2\cos\theta^2\right.\\&\left.-\frac{m^2}{\sin^2\theta}\right)S_{lm}=0.\end{aligned} \tag{7}$$

Equation (7) is for the spin-weighted hamonics[3] S_{lm} with eigenvalue λ_{lm}, where l and m are integers and $|m|\leq l$. The analytic properties of the eigenvalues and eigenfunctions of Eq. (7) have been studied.[18,19] The angular separation constant S_{lm} can be approximated[20] for small $a\omega$

$$\lambda_{lm}=\lambda_0+\lambda_1(a\omega)+\lambda_2(a\omega)^2+O((a\omega)^3), \tag{8}$$

where the coefficients are

$$\begin{aligned}&\lambda_0=l(l+1)-s(s-1),\quad \lambda_1=-\frac{2ms^2}{l(l+1)},\\&\lambda_2=f(l+1)-f(l)-1,\end{aligned} \tag{9}$$

with

$$f(l)=\frac{2(l^2-m^2)(l^2-s^2)}{(2l-1)(2l+1)}. \tag{10}$$

In this paper we only study the scalar particles, i.e., $s=0$. Hence Eq. (8) becomes

$$\lambda_{lm}=l(l+1)+\frac{1}{2}\left[1-\frac{(2m-1)(2m+1)}{(2l-1)(2l+3)}\right](a\omega)^2. \tag{11}$$

Next, we compute analytically the absorption cross section and decay rates for a massive scalar field, on the stationary Axisymmetric EMDA black hole backgroundby using the coefficients matching method. Under the low energy condition $(1/\omega\gg(M-D)$, i.e., the Compton wavelength of the scalar particle is much larger than the typical size of the black hole), following the method of Starobinsky,[21] we divide the space outside the event horizon in two regions, namely, the near-region, $r-r_+\ll 1/\omega$, and the far-region, $r-r_+\gg M_{\rm ADM}$. We solve the radial Eq. (6) in each one of these two regions. Then, we will match the near-region and the far-region solutions to obtain a solution to the problem in the overlapping region, namely the region $r_+\ll r-r_+\ll 1/\omega$.

In the near-region defined by $r-r_+\ll 1/\omega$, for $(\omega a)^2$ can be neglected, so the slow rotation black hole $(a\ll M-D)$, the radial wave equation can be written as

$$\begin{aligned}&\Sigma\partial_r(\Sigma\partial_r R)+(r_+^2-2Dr_++a^2)^2(\omega-m\Omega)^2R\\&-\lambda_{lm}\Delta R=0.\end{aligned} \tag{12}$$

To find the analytical solution of this equation, we introduce a new variable

$$z=\frac{r-r_+}{r-r_-},\qquad 0\leq z\leq 1. \tag{13}$$

The event horizon is at $z=0$. Then, one finds

$$\Sigma\partial_r=(r_+-r_-)z\partial_z. \tag{14}$$

Thus the near-region radial wave equation can be rewritten as

$$z(1-z)\partial_z^2R+(1-z)\partial_zR+\xi^2\frac{1-z}{z}R-\frac{\lambda_{lm}}{1-z}R=0, \tag{15}$$

where $\xi=(\omega-m\Omega)/(4\pi T_H)$.

Equation (15) could be transformed to standard hypergeometric[17] function form under the ansatz

$$z(1-z)\partial_z^2F+[c-(a+b+1)z]\partial_zF-abF=0, \tag{16}$$

with

$$\begin{aligned}&R=z^{-i\xi}(1-z)^{\frac{1-\sqrt{1+4\lambda_{lm}}}{2}}F(z),\\&a=-i\xi+\frac{1-\sqrt{1+4\lambda_{lm}}}{2}+\xi,\\&b=-i\xi+\frac{1-\sqrt{1+4\lambda_{lm}}}{2}-\xi,\ c=1-2i\xi.\end{aligned} \tag{17}$$

Using Ref. [17], one finds that the most general solution of the near-region equation is

$$\begin{aligned}R=&Az^{i\xi}(1-z)^{\frac{1-\sqrt{1+4\lambda_{lm}}}{2}}F(a-c+1,b-c+1,2-c,z)\\&+Bz^{-i\xi}(1-z)^{\frac{1-\sqrt{1+4\lambda_{lm}}}{2}}F(a,b,c,z).\end{aligned} \tag{18}$$

Now we need to impose some boundary conditions under the appropriate physical situations. In general, two-independent boundary conditions can be imposed in this analysis; those are based on the two equivalent pictures of probing scalar fields under the black hole background. The first one is for the classical description of black holes; a black hole can absorb the probing fields but nothing can escape from the event horizon. The second one is for the quantum description of the black hole; the asymptotic observer can

detect the quantum radiation emitted from the black hole horizon. Since both descriptions are equivalent and independent each other, we shall use the classical description, so there can be no outgoing flux across the horizon, and thus we set $A = 0$ in Eq. (18). We are now interested in the large r behavior of the ingoing near-region solution. Use the identity[17]

$$\begin{aligned} &F(a-c+1,b-c+1,2-c,z) \\ &=(1-z)^{c-a-b}\frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} \\ &\times F(1-a,1-b,c-a-b+1,1-z) \\ &+\frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \\ &\times F(a-c+1,b-c+1,-c+a+b+1,1-z), \end{aligned}$$

and the property $F(a,b,c,0) = 1$. When $r \to \infty$, $1 - z \to (r_+ - r_-)/r, \lambda_{lm} \to l(l+1)$, we obtain the large r behavior of the ingoing wave solution in the near region,

$$\begin{aligned} R \sim B\Gamma(1-2i\,\xi)\Bigg[&\frac{(r_+-r_-)^{-l}\Gamma(2l+1)}{\Gamma(l+1)\Gamma(l+1-i\frac{\omega-m\Omega}{2\pi T_H})}r^l \\ &+\frac{(r_+-r_-)^{l+1}\Gamma(-2l-1)}{\Gamma(-l)\Gamma(-l-i)}r^{-l-1}\Bigg]. \end{aligned} \tag{19}$$

In the far-region, $r - r_+ \gg (M - D)$, the effects induced by the black hole can be neglected ($a \sim 0$, $M \sim 0$, $\Delta \sim r^2$) and the radial wave Eq. (6) becomes

$$\partial_r^2(rR) + \left[\omega^2 - l(l+1)/r^2\right](rR) = 0. \tag{20}$$

The far-region solution of this equation is a linear combination of Bessel functions,[17]

$$R = r^{-1/2}\left[\alpha J_{l+1/2}(\omega r) + \beta J_{-l-1/2}(\omega r)\right]. \tag{21}$$

For large r this solution can be written as[17]

$$R \sim \sqrt{\frac{2}{\pi\omega}}\frac{1}{r}\left[\alpha\sin\left(\omega r - \frac{l\pi}{2}\right) + \beta\cos\left(\omega r + \frac{l\pi}{2}\right)\right]. \tag{22}$$

Decomposing Eq. (22), the "ingoing" and "outgoing" waves (denoted by $B_{\rm in}$ and $B_{\rm out}$, respectively) can be written as

$$B_{\rm in} = \frac{(\beta + i\alpha)e^{-i\omega r}}{2}, \quad B_{\rm out} = \frac{(\beta - i\alpha)e^{i\omega r}}{2}. \tag{23}$$

While for small r it reduces to[17]

$$R \sim \alpha\frac{(\omega/2)^{l+1/2}}{\Gamma(l+3/2)}r^l + \beta\frac{(\omega/2)^{-l-1/2}}{\Gamma(-l+1/2)}r^{-l-1}. \tag{24}$$

When $(M - D) \ll r - r_+ \ll 1/\omega$, the near-region solution and the far-region solution overlap, and thus one can match the large r near-region solution (19) with the small r far-region solution (24). This matching yields

$$B = \frac{(r_+-r_-)^l}{\Gamma(l+3/2)}\frac{\Gamma(l+1)}{\Gamma(2l+1)}\frac{\Gamma(l+1-i\,\frac{\omega-m\Omega}{2\pi T_H})}{\Gamma(1-i\,\frac{\omega-m\Omega}{2\pi T_H})}\left(\frac{\omega}{2}\right)^{l+\frac{1}{2}}\alpha, \tag{25}$$

$$\begin{aligned} \frac{\beta}{\alpha} = &\frac{\Gamma(-l+1/2)}{\Gamma(l+3/2)}\frac{\Gamma(l+1)}{\Gamma(2l+1)}\frac{\Gamma(-2l-1)}{\Gamma(-l)}\frac{\Gamma(l+1-i\frac{\omega-m\Omega}{2\pi T_H})}{\Gamma(-l-i\frac{\omega-m\Omega}{2\pi T_H})} \\ &\times\left(\frac{\omega}{2}\right)^{2l+1}(r_+-r_-)^{2l+1}. \end{aligned} \tag{26}$$

The conserved flux associated to the radial wave equation (6) is

$$\mathcal{F} = \frac{2\pi}{i}(r-r_+)(r-r_-)(\Psi^*\partial_r\Psi - \Psi\Delta\partial_r\Psi^*). \tag{27}$$

The absorption $\mathcal{A}_{lm}$ and reflection $\mathcal{R}_{lm}$ coefficients can be defined by

$$\mathcal{A}_{lm} = \frac{\mathcal{F}^{\rm in}_{\rm near}}{\mathcal{F}^{\rm in}_{\rm far}}, \quad \mathcal{R}_{lm} = \frac{\mathcal{F}^{\rm out}_{\rm near}}{\mathcal{F}^{\rm in}_{\rm far}}. \tag{28}$$

The reflection coefficient can be expressed in terms of the matched coefficients in Eqs. (23) and (26) as

$$\mathcal{R}_{lm} = \frac{\mathcal{F}^{\rm out}_{\rm near}}{\mathcal{F}^{\rm in}_{\rm far}} = \left|\frac{B_{\rm out}}{B_{\rm in}}\right|^2 = \left|\frac{\beta - i\alpha}{\beta + i\alpha}\right|^2. \tag{29}$$

Using the properties of the gamma function, we can obtain the absorption coefficient

$$\begin{aligned} \mathcal{A}_{lm} &= 1 - \mathcal{R}_{lm} \\ &= \frac{A_H(r_+-r_-)^{2l}2^{2l}(l!)^4(\omega-m\Omega)\omega^{2l+1}}{\pi[(2l)!]^2[(2l+1)!]^2} \\ &\times\prod_{n=1}^{l}\left(n^2 + \left(\frac{\omega-m\Omega}{2\pi T_H}\right)^2\right). \end{aligned} \tag{30}$$

Here $\mathcal{A}_{lm}$ is valid for both superradiantion and nonsuperradiantion under low-energy condition ($\omega M \ll 1$).

The total absorption cross-sections[6] is then given by

$$\sigma_{\rm abs} = \sum_{l=0}^{\infty}\frac{(2l+1)\pi}{\omega^2}\mathcal{A}_{lm}. \tag{31}$$

The mostly contribution to the cross section is $l = 0$ term, since T_l goes as ω^{2l+1} and as $\omega \ll 1$ by assumption. We therefore obtain

$$\sigma^{0,0}_{\rm abs} = 4\pi(r_+^2 - 2Dr_+ + a^2). \tag{32}$$

Obviously, the S wave ($l = 0$) absorption cross section is equal to the area the horizon area of the stationary axisymmetric EMDA black hole, when $D \to 0$ and $a \to 0$, we recover the Unruh's result.[6]

For the mode, $l = 1$ represents the usual partial absorption cross section for the stationary axisymmetric EMDA black hole, we find

$$\begin{aligned} \sigma^{1,m}_{\rm abs} = &\frac{1}{9}(r_+^2 - 2Dr_+ + a^2)\omega\pi(r_+-r_-)^2(\omega-m\Omega) \\ &\times\left(1 + \frac{(\omega-m\Omega)^2}{4\pi^2T_H^2}\right). \end{aligned} \tag{33}$$

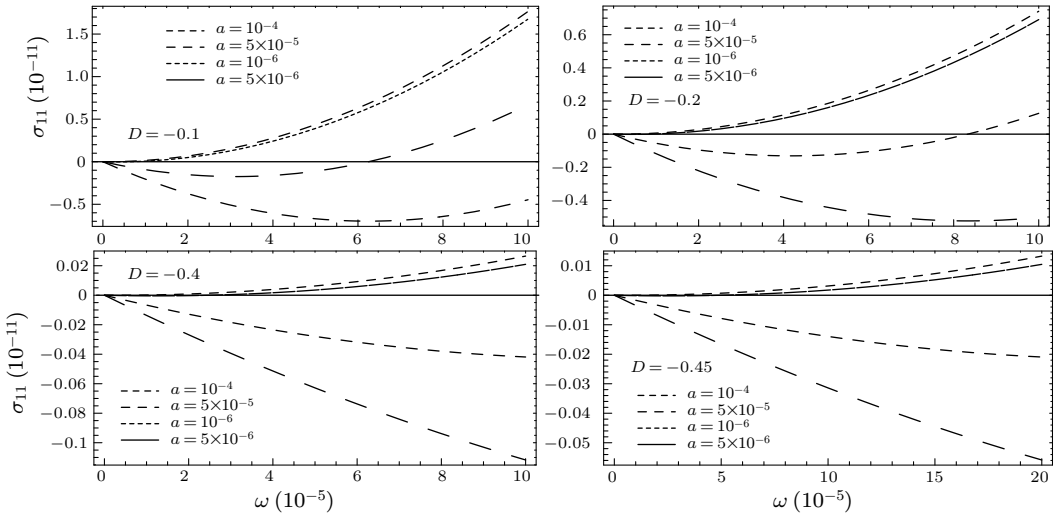

Fig. 1. The partial absorption cross section $\sigma^{1,1}_{\rm abs}$ for the stationary axisymmetric EMDA black hole versus ω with different values of the dilaton D and the rotating parameter a, for the mode $l = m = 1$.

Figure 1 shows the partial absorption cross section $\sigma^{1,1}_{\rm abs}$ for the stationary axisymmetric EMDA black hole versus ω with different values of the dilaton D and the rotating parameter a under low-energy condition ($M\omega \ll 1$). For simplicity we set $2M_{ADM} = 1$. Notice that the dilaton parameter D is negative[16] and our discussion is valid for the slow rotating black hole. Thus, we increase a from 5×10^{-6} to 10^{-4} for $D = -0.1$, -0.2, -0.4 and -0.45 respectively. From the figure we know that (1) when $\omega > m\Omega$, the partial absorption cross section $\sigma^{1,1}_{\rm abs}$ increase as the rotating parameter a or the absolute value of the dilaton D decrease, which means that the dilaton D and rotating parameter a suppress greatly the increase of the absorption cross section; (2) when $\omega < m\Omega$, the partial absorption cross section $\sigma^{1,1}_{\rm abs}$ is negative, which corresponds to superradiance. This means that the scalar wave extracts the rotational energy from the black hole and the amplitude of the scattered wave is larger than the one of the incident wave.

We can also obtain the decay rate

$$\Gamma^{\ell} = \frac{\sigma^{\ell,m}_{\rm abs}}{e^{\frac{\omega-m\Omega}{T_H}} - 1} = \frac{A_H T_H (r_+ - r_-)^{2l} 2^{2l} (l!)^4 \omega^{2l-1} (e^{-\frac{\omega-m\Omega}{T_H}})}{[(2l)!]^2[(2l+1)!]^2} \times \prod_{n=1}^{l} \left(n^2 + \left(\frac{\omega - m\Omega}{2\pi T_H}\right)^2\right). \quad (34)$$

Equation (34) indicate that the decay rate, unlike the cross section, is positive for all values of ω. For the extremal black hole, which implies $r_+ = r_-$, $T_H \to 0$, there is no Hawking radiation, however, if $a > 0$ the black hole still decays through superradiant emission.

In summary, the behavior of the massless scalar equation on the stationary axisymmetric EMDA black hole background is investigated. The absorption coefficients $\mathcal{A}_{lm}$ is given by Eq. (30) and the low energy absorption cross section of stationary axisymmetric EMDA black for massless scalar particle $\sigma_{\rm abs}$ is given by Eq. (32). It is shown that the S wave absorption cross $\sigma^{0,0}_{\rm abs}$ equals the horizon area of the black hole. This may be a universal property[9] for minimally coupled massless scalar particles. It is interesting to note that the rotating parameter a and the dilaton D suppress greatly the increase of the absorption cross section. When $\omega < m\Omega$, the partial absorption cross section $\sigma^{l,m}_{\rm abs}$ is negative, which is different from the case in the static black hole. We also calculate the decay rate and found that, unlike the cross section, it is positive for all values of ω. For the extremal black hole, the decay rate Γ^{ℓ} does not vanish and is dominated by superradiant emissions.

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