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Modulational Instability and Variable Separation Solution for a Generalized (2+1)-Dimensional Hirota Equation *

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It is demonstrated by the linear modulational instability analysis that a generalized (2+1)-dimensional Hirota equation is modulationally stable. Then, a Bäcklund transformation (BT) is obtained by means of the truncated Painlevé approach. Using the BT, the model is transformed to a system of equations, which finally leads to a special variable separation solution with arbitrary functions.

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In Ref. [1], by applying an asymptotically exact reduction method based on Fourier expansion and spatiotemporal rescaling to the Kadomtsev–Petviashvili (KP) equation, a new type of coupled nonlinear evolution equation is derived,

$$iu_t + u_{xy} + iu_{xxx} + uv - i|u|^2 u_x = 0, \tag{1}$$

$$3v_x + (|u|^2)_y = 0, \tag{2}$$

where u is complex and v is real. Taking $y = x$, Eqs. (1) and (2) reduce to

$$iu_t + u_{xx} + iu_{xxx} - \frac{1}{3}|u|^2 u - i|u|^2 u_x = 0, \tag{3}$$

which is just the (1+1)-dimensional integrable equation proposed by Hirota.[2] As pointed out in Ref. [2], the Hirota equation (3) might be further reduced to the nonlinear Schrödinger equation in a limit, and to the modified Korteweg de-Vries equation in another limit. The so-called Hirota–Maccari (HM) equations (1)–(2) are integrable due to the existence of the Lax pairs, which are also obtained from the KP equation via the reduction technique. Moreover, it is proved that the HM equations can pass the Painlevé test.[3] Some soliton solutions of Eqs. (1)–(2) have been obtained by various methods.[4–6]

To find exact solutions of nonlinear partial differential equations (PDEs) is an important but difficult task. Now, many methods have been established to find exact solutions of PDEs especially for integrable systems. Moreover, it is clear that different methods usually give different types of exact solutions.[7–11] In some cases, similar classes of solutions can be obtained by several approaches. Recently, the multi-linear variable separation approach has been proved very powerful in finding variable separation solutions for a large number of PDEs.[12–15] Later, equivalent variable separation solutions are retrieved via other methods such as the Darboux transformation method,[16–18] the entangled mapping approach and/or the modified extended tanh function method,[19] and the truncated painlevé approach.[20–22] In addition, according to the variable separation solutions, it is discovered that at least one lower dimensional arbitrary function exists in the variable separation solutions for integrable systems. As mentioned above, the HM system is proved integrable in the senses that it possesses a Lax pair and can pass the Painlevé test, consequently, its variable separation solution, if it exists, must have at least one (1+1)-dimensional arbitrary function.

In this Letter, we focus on Eqs. (1) and (2) to investigate their modulational instabilities and possible variable separation solutions. First, a nonlinear dispersion relation is derived to depict the modulational instabilities of a constant amplitude carrier wave. It is demonstrated that the HM system is modulationally stable. Then, a special variable separation solution is obtained via the truncated Painlevé approach, which includes two arbitrary functions, one is of y, t and the other is of a travelling wave variable $y - c_0 t + c_1$, as anticipated by its integrability.

Commonly, one can analyze the modulational instability of a system through the following steps: (1) find an equilibrium state of the system, (2) perturb the equilibrium state with a smaller perturbation wavenumber and frequency, (3) derive a set of equations for the small perturbation functions, which will lead to the nonlinear dispersion relation, (4) from the dispersion relation to obtain a complex frequency, revealing the growth of the amplitude modulated wave packet. From the above analysis, one can finally make a conclusion as to whether a wave under small pertur-

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bations moving along the system is stable or not.

An equilibrium state, namely, the simple and exact monochromatic wave solution of the HM equations can be easily found by making the assumption

$$u = u_0 e^{i\omega t}, \qquad v = v_0, \tag{4}$$

where constants ω, v_0 are real, and u_0 is complex. Substituting Eq. (4) into Eqs. (1)–(2), we obtain

$$\omega = v_0. \tag{5}$$

Next we make a small perturbation of the wave solution (4) in the form of

$$u = (u_0 + \epsilon u_1)e^{iv_0 t}, \qquad v = v_0 + \epsilon v_1, \tag{6}$$

where u_1 is a complex quantity and v_1 is real. Then substituting Eq. (6) into Eqs. (1)–(2), linearizing the result with respect to u_1 and v_1, writing $u_1 = r + is$, $u_0 = a + ib$, $(r, s, a, b$ are real), and then separating the real and imaginary parts of the linearized equations (the first-order terms of ϵ), we finally obtain

$$r_t + s_{xy} + r_{xxx} + bv_1 - |u_0|^2 r_x = 0, \tag{7}$$

$$s_t - r_{xy} + s_{xxx} - av_1 - |u_0|^2 s_x = 0, \tag{8}$$

and

$$3v_{1x} + 2ar_y + 2bs_y = 0. \tag{9}$$

Now inserting $r = r_0 e^{i(Kx+Ly-\Omega t)} + \text{c.c.}$, $s = s_0 e^{i(Kx+Ly-\Omega t)} + \text{c.c.}$ and $v_1 = v_{10} e^{i(Kx+Ly-\Omega t)} + \text{c.c.}$, where K, L and Ω are the perturbation wavenumbers and the frequency, respectively, and c.c. stands for the complex conjugate, into Eqs. (7)–(9) yields a dispersion law for the perturbation wave

$$\Omega^2 - P\Omega + Q = 0, \tag{10}$$

with

$$P = -2K^2(K^2 + |u_0|^2),$$

$$Q = K^6 + 2|u_0|^2 K^4 + (|u_0|^4 - L^2)K^2 - \frac{2}{3}L^2|u_0|^2. \tag{11}$$

The calculation of the discriminant quantity gives

$$\Delta \equiv P^2 - 4Q = \frac{4}{3}L^2(3K^2 + 2|u_0|^2). \tag{12}$$

It is clear that the discriminant quantity $\Delta \geq 0$ for any K and L, which means that two roots of Eq. (2), determined by $\Omega_\pm = (P \pm \sqrt{\Delta})/2$ are both real numbers. Consequently, it is concluded that the HM system (1)–(2) is modulationally stable for any wavenumbers K and L. In other words, the nonlinear waves supported by the HM system are stable under small perturbations.

To generate the variable separation solution of the HM equations (1)–(2), we introduce the following Bäcklund transformation

$$u = \frac{f}{\phi} + u_1, \quad v = \frac{v_0}{\phi^2} + \frac{v_1}{\phi} + v_2, \tag{13}$$

which is generated by truncating the Laurent series of the solutions of the HM equations at the constant level term. In Eq. (13), f and u_1 are complex functions, ϕ, v_0, v_1, v_2 are real functions, and (u_1, v_2) is a seed solution of the HM equations. Obviously, the HM system admits the following seed solution

$$u_1 = 0, \qquad v_2 = v_2(y, t). \tag{14}$$

Now substituting Eq. (13) with the seed solution (14) into Eqs. (1)–(2), collecting the coefficients of different powers of ϕ, we obtain a system of equations

$$v_{1x} = 0, \tag{15}$$

$$6\phi_x^2 - |f|^2 = 0, \tag{16}$$

$$3v_0\phi_x + |f|^2\phi_y = 0, \tag{17}$$

$$if_t + f_{xy} + if_{xxx} + fv_2 = 0, \tag{18}$$

$$3v_{0x} - 3v_1\phi_x + (|f|^2)_y = 0, \tag{19}$$

$$fv_0 - i|f|^2 f_x + 2f\phi_x\phi_y + 6if\phi_x\phi_{xx} + 6if_x\phi_x^2 = 0, \tag{20}$$

$$\begin{aligned} &fv_1 - f_x\phi_y - if\phi_{xxx} - f\phi_{xy} - 3if_{xx}\phi_x \\ &- f_y\phi_x - 3if_x\phi_{xx} - if\phi_t = 0, \end{aligned} \tag{21}$$

determining the unknowns f, ϕ, v_0, v_1 and v_2.

From Eq. (16), we have

$$|f|^2 = 6\phi_x^2. \tag{22}$$

From Eq. (17) with Eq. (22), we obtain

$$v_0 = -2\phi_x\phi_y. \tag{23}$$

Substituting Eqs. (22) and (23) into Eq. (20) yields

$$\phi_x\phi_{xx} = 0, \tag{24}$$

which has a general solution in the form of

$$\phi = Fx + G, \tag{25}$$

with $F \equiv F(y, t)$ and $G \equiv G(y, t)$ being the integration functions of the indicated variables.

Due to Eq. (22) with Eq. (25), the complex function f can be assumed to have the following general expression

$$f = \sqrt{6}Fe^{i(r+s)}, \tag{26}$$

where $r \equiv r(x, t)$ and $s \equiv s(y, t)$ are real functions of the indicated arguments.

Now substituting Eqs. (25) and (26) into Eq. (21) and then separating the real and imaginary parts, we obtain

$$v_1 - 2F_{1y} + 3Fr_{xx} = 0, \tag{27}$$

$$r_x G_y + G_t + Fs_y - 3Fr_x^2 + (F_t + F_y r_x)x = 0. \tag{28}$$

From Eqs. (15) and (27), we have $r_{xxx} = 0$, whose general solution reads

$$r = r_2 x^2 + r_1 x + r_0, \tag{29}$$

where r_i $(i = 0, 1, 2)$ are the integration functions of t. Then Eq. (27) with (29) gives

$$v_1 = 2F_{1y} - 6Fr_2. \tag{30}$$

Inserting Eqs. (25), (26) and (29) into Eq. (18), separating the real and imaginary parts and then solving the obtained equations, we find

$$r_2 = 0,\ r_1 = c_0,\ F = F(y - c_0 t + c_1), \tag{31}$$

and

$$v_2 = c_0 s_y + s_t + r_{0t} - c_0^3, \tag{32}$$

where c_0 and c_1 are arbitrary constants.

From Eq. (28) with the help of Eqs. (29) and (31), we obtain

$$s = -\int \frac{c_0 G_y + G_t}{F} dy + 3c_0^2 y. \tag{33}$$

Finally, using Eqs. (23), (26) and (30) with (31), it is easy to find that Eq. (19) is satisfied identically.

Therefore, the final variable separation solution of the HM equations (1)–(2) reads

$$u = \frac{\sqrt{6}F}{Fx + G} \exp\left\{ i\left(c_0 x + r_0 + 3c_0^2 y - \int \frac{c_0 G_y + G_t}{F} dy \right) \right\}, \tag{34}$$

and

$$v = -\frac{2F(F_y x + G_y)}{(Fx + G)^2} + \frac{2F_y}{Fx + G} - \frac{c_0(c_0 G_y + G_t)}{F} + r_{0t} + 2c_0^3 - \int \frac{F(c_0 G_{yt} + G_{tt}) - F_t(c_0 G_y + G_t)}{F^2} dy, \tag{35}$$

where $F \equiv F(y - c_0 t + c_1)$, $G \equiv G(y, t)$, $r_0 \equiv r_0(t)$ are arbitrary functions of the indicated variables, c_0 and c_1 are arbitrary constants. It is remarkable that due to the arbitrary functions F and G of the indicated variables, the new variable separation solution obviously describes non-traveling waves, which is different from those traveling wave solutions obtained in Ref. [4–6]. Moreover, as manifested in Refs. [12–15,18–22], by suitably adjusting the arbitrary functions, various interesting nonlinear coherent structures can be excited on the basis of the variable separation solutions. It is noted that the solutions (34) and (35) might blow up, and thus explosive waves will be readily generated. Another type of wave, namely, peaked waves, can be excited easily by choosing F as a piecewise function in the $x - y$ plane. For instance, if F is taken in the form

$$F = \begin{cases} -\mathrm{sech}(y - t)^2 - \frac{1}{2}\mathrm{sech}(y - 8 + t)^2, & x < 0, \\ \mathrm{sech}(y - t)^2 + \frac{1}{2}\mathrm{sech}(y - 8 + t)^2), & x \geq 0, \end{cases} \tag{36}$$

whose structure is displayed in Fig. 1, and $G = 1 + \mathrm{sech}(y + t + 2)^2$, then the quantity $|u|^2$ describes a two-peakon wave, and v_x is a two-dipole-peakon solution. Figure 2 shows the interaction between two peakons.

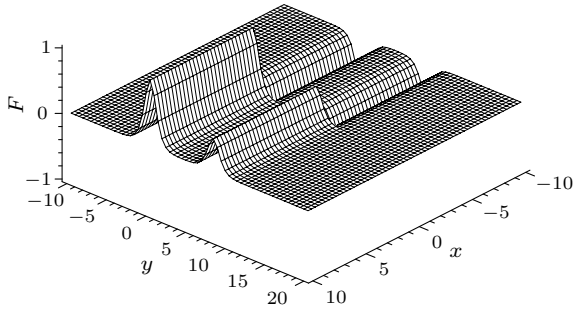

Fig. 1. Profile of F expressed by Eq. (36) at time $t = 0$.

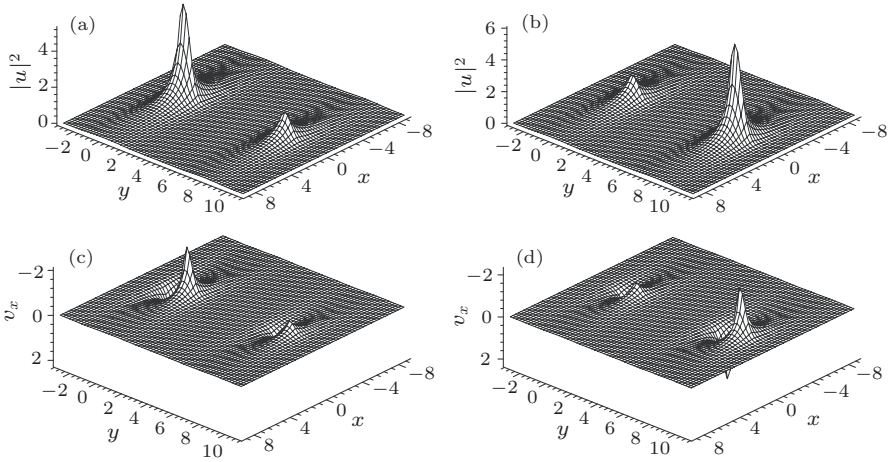

Fig. 2. Interaction between two peakons expressed by $|u|^2$ and v_x given by Eqs. (34) and (35) with Eq. (36) and $G = 1 + \mathrm{sech}(y + t + 2)^2$ at times (a,c) $t = 0$ before the interaction, and (b,d) $t = 8$ after the interaction, respectively.

In summary, we have first investigated the modulational instability of the generalized (2+1)-dimensional Hirota equation, also called the Hirota–Maccari (HM) equations. It is demonstrated that the HM system is modulationally stable for any wavenumbers. In other words, nonlinear waves governed by the HM equations are stable under small perturbations. We have then obtained a variable separation solution of the HM equations. As expected from the integrability of the HM system, (1+1)-dimensional dimensional arbitrary function does enter its variable separation solution, and thus various nonlinear excitations and interaction behaviors can be constructed.

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