CHIN. PHYS. LETT. Vol. 27, No. 1 (2010) 010503

A Passivity Based Synchronization for Chaotic Behavior in Nonlinear Bloch Equations *

Choon Ki Ahn**

Faculty of the Division of Electronic and Control Engineering, Wonkwang University, 344-2 Shinyong-dong, Iksan 570-749, Korea

(Received 3 August 2009)

We propose a new passivity based synchronization scheme for chaotic behavior in nonlinear Bloch equations. Based on the Lyapunov theory and linear matrix inequality (LMI) approach, for the first time, the passivity based controller is presented to guarantee stable synchronization. The proposed controller can be obtained by solving a convex optimization problem represented by an LMI. A numerical example is given to demonstrate the effectiveness of the proposed synchronization scheme.

PACS: 05.45.Gg, 05.45.−a DOI: 10.1088/0256-307X/27/1/010503

Synchronization is a fundamental phenomenon that enables coherent behavior in coupled dynamical systems. During the last two decades, synchronization in chaotic dynamic systems has received a great deal of interest among scientists from various research fields since Pecora and Carroll[1] introduced a method to synchronize two identical chaotic systems with different initial conditions. It has been widely explored in a variety of fields including physical, chemical and ecological systems.[2] In the literature, various synchronization schemes, such as variable structure control,[3] OGY method,[4] parameters adaptive control,[5,6] observer-based control,[7] active control,[8,9] time-delay feedback approach,[10] backstepping design technique,[11,12] and complete synchronization,[13] have been successfully applied to the chaos synchronization.

The dynamics of an ensemble of spins usually described by the nonlinear Bloch equation is very important for the understanding of the underlying physical process of nuclear magnetic resonance. The basic process can be viewed as the combination of a precession about a magnetic field and of a relaxation process. Abergel[14] demonstrated that the set of nonlinear Bloch equations would admit chaotic solutions for a certain set of numerical values assigned to the system constants and initial conditions. Ucar *et al.*[15] extended the calculation of Abergel[14] and demonstrated that an active control method can synchronize two of these nonlinear Bloch equations. Recently, some control schemes, such as adaptive control[16] and stability criterion method,[17] were proposed for synchronizing chaotic behavior in nonlinear Bloch equations.

The concept of passivity for nonlinear systems has attracted new interest in nonlinear system control. The main idea of passivity theory is that the passive properties for systems can keep the system internal stability.[18] It is well-known that the passivity theory plays an important role in designing the asymptotically stabilizing controller for nonlinear systems. Wen[19] applied the passivity technique to design the controller, whose structure is of linear feedback form, to control the Lorenz system. In Ref. [20], the passivity based controller was proposed for the chaotic Lü system. Wei *et al.*[21] proposed the passivity based control for chaotic oscillations in the power system. Wang *et al.*[22] also applied this technique to design a controller to control a unified chaotic system to zero and any desired equilibrium. Recently, the passivity based control for the nuclear spin generator chaotic system was proposed in Ref. [23]. To the best of our knowledge, however, for the passivity based synchronization of chaotic behavior in nonlinear Bloch equations, there is no result in the literature so far, which still remains open and challenging.

In this Letter, a new controller for the passivity-based synchronization of chaotic behavior in nonlinear Bloch equations is proposed. This controller is a new contribution to the topic of chaos synchronization for nonlinear Bloch equations. Theoretical proof reveals that the use of the proposed control could make the closed-loop error system passive and also asymptotically stable. Based on the Lyapunov method and linear matrix inequality (LMI) approach, the passivity-based synchronization problem of chaotic behavior in nonlinear Bloch equations is formulated in terms of an LMI. The LMI can be solved efficiently by using recently developed convex optimization algorithms.[24]

Basic concept of passivity. Consider the following differential equation

$$\dot{X}(t) = F(X(t)) + G(X(t))U(t), \tag{1}$$

$$Y(t) = H(X(t)), \tag{2}$$

where $X(t) \in \Re^n$ is the state variable, $U(t) \in \Re^m$ is the external input, and $Y(t) \in \Re^m$ is the output. F and G are smooth vector fields and H is a smooth

*Supported by the Grant of the Korean Ministry of Education, Science and Technology (The Regional Core Research Program/Center for Healthc Technology Development).

**Email: hironaka@wonkwang.ac.kr

© 2010 Chinese Physical Society and IOP Publishing Ltd

mapping. Without loss of generality, we suppose that the vector field F has at least one equilibrium point. The notion of passivity can be described in the following.

Definition 1. If there exist a nonnegative constant β and a positive semi-definite function $S(X(t))$ such that

$$\int_0^t U^T(\tau)Y(\tau)d\tau + \beta \geq \int_0^t S(X(\tau))d\tau, \ \forall\, t \geq 0, \quad (3)$$

the system (1)–(2) is said to be passive from the external input $U(t)$ to the output $Y(t)$.

The physical meaning of passive system is that the energy of the nonlinear system (1)–(2) can be increased only through the supply from the external source. In other words, a passive system cannot store more energy than it is supplied. The passive system is a naturally stable system. The passive system exploits the input-output relationship based on energy-related considerations to analyze stability properties.

We will use the passivity technique to control nonlinear Bloch equations. The controller proposed in this study is based on an LMI. Using the passive method, it is very easy to prove the stability of the closed-loop system.

Nonlinear Bloch Equations. In dimensionless units, the dynamic model of nonlinear modified Bloch equations with feedback field[14] is given by

$$\begin{aligned}
\dot{x}_d(t) &= \delta y_d(t) + \lambda z_d(t)(x_d(t)\sin\psi - y_d(t)\cos\psi) \\
&\quad - x_d(t)/\tau_2, \\
\dot{y}_d(t) &= -\delta x_d(t) - z_d(t) + \lambda z_d(t)(x_d(t)\cos\psi \\
&\quad + y_d(t)\sin\psi) - y_d(t)/\tau_2, \\
\dot{z}_d(t) &= y_d(t) - \lambda\sin\psi(x_d^2(t) + y_d^2(t)) \\
&\quad - [z_d(t) - 1]/\tau_1, \qquad (4)
\end{aligned}$$

where δ, λ and ψ are the system parameters and τ_1 and τ_2 are the longitudinal time and transverse relaxation times, respectively. The subscript d indicates that the system will be considered as the drive (or master) system. Abergel has extensively investigated the dynamics of the system (4) for a fixed subset of the system parameters (δ, λ, τ_1, τ_2) and for a space area range of the radiation damping feedback ψ.[14] In particular, the regions of the radiation damping feedback ψ that would admit chaotic behavior were obtained. For details of other dynamic properties of the system (4), refer to Refs. [14,15].

The synchronization problem of system (4) is considered using the drive-response configuration. The system (4) is considered as the drive system. According to the drive-response concept, the controlled response (or slave) system is given by

$$\begin{aligned}
\dot{x}_r(t) &= \delta y_r(t) + \lambda z_r(t)(x_r(t)\sin\psi - y_r(t)\cos\psi) \\
&\quad - \frac{x_r(t)}{\tau_2} + u_1(t), \\
\dot{y}_r(t) &= -\delta x_r(t) - z_r(t) + \lambda z_r(t)(x_r(t)\cos\psi \\
&\quad + y_r(t)\sin\psi) - \frac{y_r(t)}{\tau_2} + u_2(t), \\
\dot{z}_r(t) &= y_r(t) - \lambda\sin\psi(x_r^2(t) + y_r^2(t)) \\
&\quad - \frac{z_r(t) - 1}{\tau_1} + u_3(t), \qquad (5)
\end{aligned}$$

where $u_1(t)$, $u_2(t)$, and $u_3(t)$ are the nonlinear controllers. Define the synchronization error as

$$\begin{aligned}
&e_1(t) = x_r(t) - x_d(t), \quad e_2(t) = y_r(t) - y_d(t), \\
&e_3(t) = x_r(t) - z_d(t). \qquad (6)
\end{aligned}$$

Then we obtain the following synchronization error system:

$$\begin{aligned}
\dot{e}_1(t) &= \delta e_2(t) - \frac{1}{\tau_2}e_1(t) + \lambda z_r(t)[x_r(t)\sin\psi \\
&\quad - y_r(t)\cos\psi] - \lambda z_d(t)(x_d(t)\sin\psi \\
&\quad - y_d(t)\cos\psi) + u_1(t), \\
\dot{e}_2(t) &= -\delta e_1(t) - e_3(t) - \frac{1}{\tau_2}e_2(t) + \lambda z_r(t)[x_r(t)\cos\psi \\
&\quad + y_r(t)\sin\psi] - \lambda z_d(t)(x_d(t)\cos\psi \\
&\quad + y_d(t)\sin\psi) + u_2(t), \\
\dot{e}_3(t) &= e_2(t) - \frac{1}{\tau_1}e_3(t) - \lambda\sin\psi(x_r^2(t) + y_r^2(t)) \\
&\quad + \lambda\sin\psi(x_d^2(t) + y_d^2(t)) + u_3(t), \qquad (7)
\end{aligned}$$

which can be rewritten as

$$\dot{e}(t) = Ae(t) + f(t) + u(t), \qquad (8)$$

where $e(t)$, $u(t)$, $A(t)$, and $f(t)$ are defined by

$$e(t) \triangleq \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}, \quad u(t) \triangleq \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}, \quad A \triangleq \begin{bmatrix} -\frac{1}{\tau_2} & \delta & 0 \\ -\delta & -\frac{1}{\tau_2} & -1 \\ 0 & 1 & -\frac{1}{\tau_1} \end{bmatrix},$$

$$f(t) \triangleq \begin{bmatrix} \lambda z_r(t)(x_r(t)\sin\psi - y_r(t)\cos\psi) - \lambda z_d(t)(x_d(t)\sin\psi - y_d(t)\cos\psi) \\ \lambda z_r(t)(x_r(t)\cos\psi + y_r(t)\sin\psi) - \lambda z_d(t)(x_d(t)\cos\psi + y_d(t)\sin\psi) \\ -\lambda\sin\psi(x_r^2(t) + y_r^2(t)) + \lambda\sin\psi(x_d^2(t) + y_d^2(t)) \end{bmatrix}.$$

Main Results. The LMI problem for achieving the passivity based synchronization is presented in the following theorem.

Theorem 2. For a given $Q = Q^T > 0$, if there exist $X = X^T > 0$ and Y such that

$$\begin{bmatrix} AX + XA^T + Y + Y^T & X \\ X & -Q^{-1} \end{bmatrix} < 0, \tag{9}$$

then the error system (8), under the control

$$u(t) = YX^{-1}\begin{bmatrix} x_r(t) - x_d(t) \\ y_r(t) - y_d(t) \\ z_r(t) - z_d(t) \end{bmatrix} - \begin{bmatrix} \lambda z_r(t)(x_r(t)\sin\psi - y_r(t)\cos\psi) - \lambda z_d(t)(x_d(t)\sin\psi - y_d(t)\cos\psi) \\ \lambda z_r(t)(x_r(t)\cos\psi + y_r(t)\sin\psi) - \lambda z_d(t)(x_d(t)\cos\psi + y_d(t)\sin\psi) \\ -\lambda\sin\psi(x_r^2(t) + y_r^2(t)) + \lambda\sin\psi(x_d^2(t) + y_d^2(t)) \end{bmatrix} + v(t), \tag{10}$$

where $v(t)$ is an external input, is passive from the external input $v(t)$ to the output $y(t)$ which is defined as

$$y(t) \triangleq 2Pe(t). \tag{11}$$

Proof. The closed-loop error system with the control $u(t) = Ke(t) - f(t) + v(t)$, where $K \in \Re^{3\times 3}$ is the gain matrix of the control $u(t)$, can be written as

$$\dot{e}(t) = (A + K)e(t) + v(t). \tag{12}$$

Consider a Lyapunov function

$$V(e(t)) = e^T(t)Pe(t), \tag{13}$$

where $P = P^T > 0$. Its time derivative along the trajectory of (12) is

$$\begin{aligned} &\dot{V}(e(t)) = \dot{e}(t)^T Pe(t) + e^T(t)P\dot{e}(t) \\ &= e^T(t)[A^TP + PA + PK + K^TP]e(t) + 2e^T(t)Pv(t) \\ &= e^T(t)[A^TP + PA + PK + K^TP]e(t) + y^T(t)v(t). \end{aligned}$$

If the following matrix inequality is satisfied

$$A^TP + PA + PK + K^TP + Q < 0, \tag{14}$$

we have

$$\dot{V}(e(t)) < -e^T(t)Qe(t) + y^T(t)v(t). \tag{15}$$

Integrating both sizes of Eq. (15) from 0 to t gives

$$\begin{aligned} V(e(t)) - V(e(0)) < &-\int_0^t e^T(\tau)Qe(\tau)d\tau \\ &+ \int_0^t y^T(\tau)v(\tau)d\tau. \end{aligned} \tag{16}$$

Let $\beta = V(e(0))$. Since $V(e(t)) \geq 0$,

$$\begin{aligned} \int_0^t y^T(\tau)v(\tau)d\tau + \beta &> \int_0^t e^T(\tau)Qe(\tau)d\tau + V(e(t)) \\ &\geq \int_0^t e^T(\tau)Qe(\tau)d\tau. \end{aligned} \tag{17}$$

The relation (17) satisfies the passivity definition (3). Therefore, the error system (8) is rendered to be passive from the external input $v(t)$ to the output $y(t)$ under the feedback control $u(t) = Ke(t) - f(t) + v(t)$.

From Schur complement, the matrix inequality (14) is equivalent to

$$\begin{bmatrix} A^TP + PA + PK + K^TP & I \\ I & -Q^{-1} \end{bmatrix} < 0. \tag{18}$$

Pre- and post-multiplying (18) by $diag(P^{-1}, I)$ and introducing change of variables such as $X = P^{-1}$ and $Y = KP^{-1}$, Eq. (18) is equivalently changed into the LMI (9). Then the gain matrix of the control $u(t)$ is given by $K = YX^{-1}$. This completes the proof.

Corollary 1 *(zero-input error response).* If the external input $v(t)$ is zero, the closed-loop error system is asymptotically stable.

Proof. When $v(t) = 0$, we obtain

$$\dot{V}(e(t)) < -e^T(t)Qe(t) \leqq 0 \tag{19}$$

from Eq. (15). This guarantees

$$\lim_{t\to\infty} e(t) = 0 \tag{20}$$

from Lyapunov theory. This completes the proof.

Corollary 3 *(nonzero-input error response).* If the external input $v(t)$ is selected to be

$$v(t) = -\mu y(t) = -2\mu Pe(t), \quad \mu > 0, \tag{21}$$

the closed-loop error system is asymptotically stable.

Proof. For $v(t) = -\mu y(t)$, the time derivative of $V(e(t))$ satisfies

$$\dot{V}(e(t)) < -e^T(t)Qe(t) - \mu y^T(t)y(t) \leq 0, \tag{22}$$

from Eq. (15). This guarantees the asymptotical stability from Lyapunov theory. This completes the proof.

Remark. The LMI problem given in Theorem 1 is to determine whether the solution exists or not. It is called the feasibility problem. The LMI problem can be solved efficiently by using recently developed convex optimization algorithms.[24] In this study, in order to solve the LMI problem, we utilize MATLAB LMI Control Toolbox,[26] which implements state-of-the-art interior-point algorithms.

Numerical example. To verify and demonstrate the effectiveness of the proposed method, we discuss the simulation result for synchronizing nonlinear Bloch equations under different initial conditions. For the numerical simulation, the system parameters ψ, τ_1, τ_2, δ, and λ are fixed as 0.173, 5, 2.5, -0.4π, and 35, respectively, so that the Bloch equations exhibit

a chaotic behavior. Let $Q = I$, where $I \in \Re^{3\times 3}$ is an identity matrix. Applying Theorem 1 to the Bloch equations yields

$$X = \begin{bmatrix} -1.1000 & -6.4948 & 20.0303 \\ 6.4948 & -1.1000 & -0.9240 \\ -20.0303 & 0.9240 & -1.3000 \end{bmatrix},$$
$$Y = \begin{bmatrix} 1.2778 & 0.0000 & 0.0000 \\ 0.0000 & 1.2778 & 0.0000 \\ 0.0000 & 0.0000 & 1.2778 \end{bmatrix}. \quad (23)$$

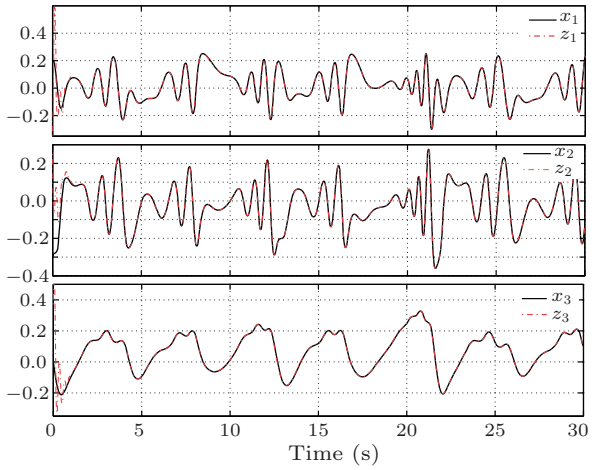


Fig. 1. State trajectories.

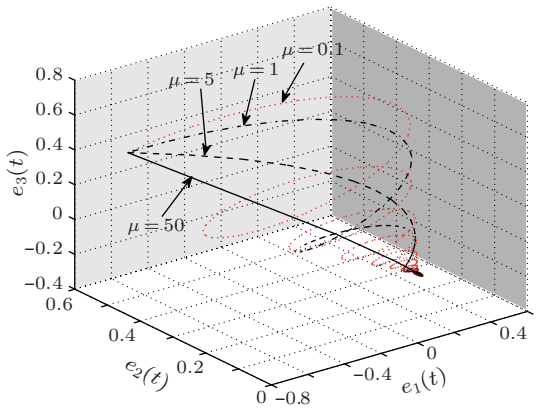


Fig. 2. Synchronization error trajectories.

Figure 1 shows the state trajectories for drive and response systems when the parameter μ of the external input (21) is given by $\mu = 1$ and the initial conditions are given by

$$\begin{bmatrix} x_d(0) \\ y_d(0) \\ z_d(0) \end{bmatrix} = \begin{bmatrix} 0.2 \\ -0.3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_r(0) \\ y_r(0) \\ z_r(0) \end{bmatrix} = \begin{bmatrix} -0.4 \\ 0.25 \\ 0.37 \end{bmatrix}. \quad (24)$$

From Fig. 1, it can be seen that drive and response systems are indeed achieving chaos synchronization. The simulation result in Fig. 2 shows the synchronization error trajectories for different values of the parameter μ. It is found from Fig. 2 that the parameter μ only influences the transient response and the larger μ gives the better response.

In summary, we have proposed a new passivity based synchronization scheme for chaotic behavior in nonlinear Bloch systems. Based on the Lyapunov theory and LMI approach, the proposed method guarantees to make the closed-loop error system passive and also asymptotically stable. Furthermore, a numerical simulation is given to illustrate the effectiveness of the proposed scheme.

References

[1] Pecora L M and Carroll T L 1996 *Phys. Rev. Lett.* **64** 821
[2] Chen G and Dong X 1998 *From Chaos to Order* (Singapore: World Scientific)
[3] Wang C C and Su J P 2004 *Chaos, Solitons and Fractal* **20** 967
[4] Ott E, Grebogi C and Yorke J A 1990 *Phys. Rev. Lett.* **64** 1196
[5] Park J H 2005 *Int. J. Nonlinear Sci. Numer. Simul.* **6** 201
[6] Wang Y, Guan Z H and Wang H O 2003 *Phys. Lett.* A **312** 34
[7] Yang X S and Chen G 2002 *Chaos, Solitons and Fractals* **13** 1303
[8] Bai E and Lonngen K 1997 *Phys. Rev.* E **8** 51
[9] Bai E W and Lonngren K E 2000 *Chaos, Solitons and Fractals* **11** 1041
[10] Park J H and Kwon O M 2005 *Chaos, Solitons and Fractals* **23** 445
[11] Wu X and Lu J 2003 *Chaos, Solitons and Fractals* **18** 721
[12] Hu J, Chen S and Chen L 2005 *Phys. Lett.* A **339** 455
[13] Zhan M, Wang X, Gong X, Wei G and Lai C 2003 *Phys. Rev.* E **68** 6208
[14] Abergel D 2002 *Phys. Lett.* A **302** 17
[15] Ucar A, Lonngren K E and Bai E W 2003 *Phys. Lett.* A **314** 96
[16] Park J H 2006 *Chaos, Solitons and Fractals* **27** 357
[17] Ghosh D, Chowdhury A R and Saha P 2008 *Commun. Nonlinear Sci. Numer. Simulat.* **13** 1461
[18] Willems J C 1972 *Arch. Rational Mech. Anal.* **45** 321
[19] Wen Y 1999 *IEEE Trans. Circuits Syst.* I **46** 876
[20] Kemih K, Filali S, Benslama M and Kimouche M 2006 *Int. J. Innovat. Comput. Inform. Control* **2** 331
[21] Wei D Q and Luo X S 2007 *Chaos, Solitons and Fractals* **31** 665
[22] Wang F and Liu C 2007 *Physica* D **225** 55
[23] Kemih K 2008 *Chaos, Solitons and Fractals* (Preprint doi:10.1016/j.chaos.2008.07.042)
[24] Boyd S, Ghaoui L E, Feron E and Balakrishinan V 1994 *Linear Matrix Inequalities in Systems and Control Theory* (Philadelphia: SIAM)
[25] Byrnes C I, Isidori A and Willem J C 1991 *IEEE Trans. Automat. Contr.* **36** 1228
[26] Gahinet P, Nemirovski A, Laub A J and Chilali M 1995 *LMI Control Toolbox* (Natick: Mathworks)