CHIN. PHYS. LETT. Vol. 26, No. 9 (2009) 090201

Multi-Soliton Solutions of the Levi Equations

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(Received 30 April 2009)

The multisoliton solutions of the Levi equations are derived with the Hirota method and Wronskian technique respectively.

PACS: 02.30.Ik, 02.30.Jr, 05.45.Yv

Up to now, there have been many powerful methods to find exact solutions for nonlinear evolution equations, such as the inverse scattering transformation,[1] the Bäcklund transformation,[2] the Darboux transformation,[3] the Hirota's bilinear method,[4] and the Wronskian technique.[5–13] The Hirota method can transform certain nonlinear evolution equations into the bilinear forms, which are relatively easy to solve. However, the verification of the N-soliton solution is very difficult. The Wronskian technique makes the verification of the N-soliton solution easier by direct substitution because differentiation of a Wronskian determinant can be easily handled and its derivatives take similar compact forms. Satsuma firstly gave the Wronskian representation of the soliton solution to the KdV equation. Then Freeman and Nimmo developed the technique. Sirianunpiboon *et al.*[14] generalized the Wronskian technique in order that the Wronskian can generate rational solutions and their interaction solutions with multi-solitons in 1988. Later, Matveev[15] introduced the generalized Wronskian to obtain positons for the KdV equation. In 2005, Ma *et al.*[16] obtained complexitons for the KdV by assuming that the coefficient matrix of the Wronskian entries has complex eigenvalues. Recently, Chen *et al.*[17] extended the lower triangular matrix equation satisfied by Wronskian entries to the arbitrary matrix equation and obtained the rational solutions, Matveev solutions, complexitons and interaction solutions of the AKNS equations.

In this Letter, we would like to consider the Levi equations by Hirota method and Wronskian technique, respectively. By introducing a suitable transformation of the potential function and writing soliton equations into bilinear from, the 1-soliton, 2-soliton and 3-soliton of the Levi equations can be derives simply. Further, N-soliton formula can be conjectured. Also we give the soliton solutions in double Wronskian form for the Levi equations.

Considering the Levi equations

$$u_t = -u_{xx} + 2uu_x + 2v_x,$$
$$v_t = v_{xx} + 2(uv)_x. \tag{1}$$

Through the transformation

$$u = -\big[\ln(g/f)\big]_x, \quad v = (\ln f)_{xx}, \tag{2}$$

then Eq. (1) can be transformed into the bilinear form

$$(D_t - D_x^2) f \cdot g = 0, \tag{3}$$
$$D_x^2 f \cdot f = 2gh, \tag{4}$$
$$D_t D_x f \cdot f = 2 D_x h \cdot g, \tag{5}$$

where D is the well-known Hirota's bilinear operator defined by[4]

$$D_t^m D_x^n a \cdot b = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n a(x,t) \cdot b(x',t') \,|_{x=x',t=t'}, m, n = 0, 1, \ldots$$

we expand f and g into power series of a small parameter ϵ as

$$f = 1 + \sum_{j=1}^{\infty} f^{(2j)} \varepsilon^{2j}, \quad g = \sum_{j=1}^{\infty} g^{(2j-1)} \varepsilon^{2j-1},$$
$$h = \sum_{j=1}^{\infty} h^{(2j-1)} \varepsilon^{2j-1}, \tag{6}$$

substituting Eq. (6) into Eqs. (3)–(5) and equating coefficients of ϵ yield

$$\begin{aligned} &g_t^{(1)} = -g_{xx}^{(1)}, \; g_t^{(3)} + g_{xx}^{(3)} = (D_t - D_x^2) f^{(2)} \cdot g^{(1)}, \\ &g_t^{(5)} + g_{xx}^{(5)} = (D_t - D_x^2)(f^{(2)} \cdot g^{(3)} + f^{(4)} \cdot g^{(1)}), \\ &\qquad \cdots \end{aligned} \tag{7}$$

$$\begin{aligned} &f_{xx}^{(2)} = g^{(1)} h^{(1)}, \\ &f_{xx}^{(4)} = -\frac{1}{2} D_x^2 f^{(2)} \cdot f^{(2)} + g^{(1)} h^{(3)} + g^{(3)} h^{(1)}, \\ &f_{xx}^{(6)} = -\frac{1}{2} D_x^2 (f^{(2)} \cdot f^{(4)} + f^{(4)} \cdot f^{(2)}) + g^{(1)} h^{(5)} \\ &\qquad + g^{(3)} h^{(3)} + g^{(5)} h^{(1)}, \\ &\qquad \cdots \end{aligned} \tag{8}$$

$$\begin{aligned} &f_{tx}^{(2)} = D_x h^{(1)} \cdot g^{(1)}, \\ &f_{tx}^{(4)} = -\frac{1}{2} D_t D_x f^{(2)} \cdot f^{(2)} + D_x (h^{(1)} \cdot g^{(3)} + h^{(3)} \cdot g^{(1)}), \end{aligned}$$

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$$f_{tx}^{(6)} = -\frac{1}{2}D_tD_x(f^{(2)}\cdot f^{(4)} + f^{(4)}\cdot f^{(2)}) + D_x(h^{(1)}\cdot g^{(5)} + h^{(3)}\cdot g^{(3)} + h^{(5)}\cdot g^{(1)}), \quad \cdots \tag{9}$$

Taking

$$g^{(1)} = e^{\xi_1}, \quad h^{(1)} = e^{\eta_1}, \quad \xi_1 = k_1x + w_1t + \xi_1^{(0)}, \quad \eta_1 = l_1x + m_1t + \eta_1^{(0)}, \ w_1 = -k_1^2, m_1 = l_1^2. \tag{10}$$

From Eqs. (7)–(9), we have

$$f^{(2)} = e^{\xi_1+\eta_1+\theta_{13}}, \quad f^{(2j)} = 0, \quad g^{(2j-1)} = h^{(2j-1)} = 0, \quad j = 2, 3, \ldots \tag{11}$$

where $e^{\theta_{13}} = \frac{1}{(k_1+l_1)^2}$.
Thus the one-soliton solution for the Levi equations is

$$u = \frac{l_1e^{\frac{\xi_1+\eta_1+\theta_{13}}{2}} - k_1e^{-\frac{\xi_1+\eta_1+\theta_{13}}{2}}}{2}\operatorname{sech}\frac{\xi_1+\eta_1+\theta_{13}}{2}, \quad v = \frac{(k_1+l_1)^2}{4}\operatorname{sech}^2\frac{\xi_1+\eta_1+\theta_{13}}{2}. \tag{12}$$

Similar to the one-soliton solution, if we take

$$g^{(1)} = e^{\xi_1} + e^{\xi_2}, \ \xi_i = k_ix + w_it + \xi_i^{(0)}, \quad w_i = -k_i^2, \quad i = 1, 2, \tag{13}$$

$$h^{(1)} = e^{\eta_1} + e^{\eta_2}, \ \eta_i = l_ix + m_it + \eta_i^{(0)}, \quad m_i = l_i^2, \quad i = 1, 2, \tag{14}$$

then

$$f^{(2)} = e^{\xi_1+\eta_1+\theta_{13}} + e^{\xi_1+\eta_2+\theta_{14}} + e^{\xi_2+\eta_1+\theta_{23}} + e^{\xi_2+\eta_2+\theta_{24}}, \tag{15}$$

$$g^{(3)} = e^{\xi_1+\xi_2+\eta_1+\theta_{12}+\theta_{13}+\theta_{23}} + e^{\xi_1+\xi_2+\eta_2+\theta_{12}+\theta_{14}+\theta_{24}}, \tag{16}$$

$$h^{(3)} = e^{\xi_1+\eta_1+\eta_2+\theta_{13}+\theta_{14}+\theta_{34}} + e^{\xi_2+\eta_1+\eta_2+\theta_{23}+\theta_{24}+\theta_{34}}, \tag{17}$$

$$f^{(4)} = e^{\xi_1+\xi_2+\eta_1+\eta_2+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{23}+\theta_{24}+\theta_{34}}, \tag{18}$$

$$f^{(2j)} = 0, g^{(2j-1)} = h^{(2j-1)} = 0, j = 3, 4, \cdots$$

where $e^{\theta_{14}} = \frac{1}{(k_1+l_2)^2}$, $e^{\theta_{23}} = \frac{1}{(k_2+l_1)^2}$, $e^{\theta_{24}} = \frac{1}{(k_2+l_2)^2}$, $e^{\theta_{12}} = (k_1-k_2)^2$, $e^{\theta_{34}} = (l_1-l_2)^2$. Therefore the two-soliton solution is obtained,

$$u = -\left[\ln(g_2/f_2)\right]_x, \quad v = (\ln f_2)_{xx},$$
$$f_2 = 1 + e^{\xi_1+\eta_1+\theta_{13}} + e^{\xi_1+\eta_2+\theta_{14}} + e^{\xi_2+\eta_1+\theta_{23}} + e^{\xi_2+\eta_2+\theta_{24}} + e^{\xi_1+\xi_2+\eta_1+\eta_2+\theta_{12}+\theta_{13}+\theta_{14}+\theta_{23}+\theta_{24}+\theta_{34}},$$
$$g_2 = e^{\xi_1} + e^{\xi_2} + e^{\xi_1+\xi_2+\eta_1+\theta_{12}+\theta_{13}+\theta_{23}} + e^{\xi_1+\xi_2+\eta_2+\theta_{12}+\theta_{14}+\theta_{24}}. \tag{19}$$

If we take

$$g^{(1)} = e^{\xi_1} + e^{\xi_2} + e^{\xi_3}, \quad \xi_i = k_ix + \omega_it + \xi_i^{(0)}, \quad \omega_i = -k_i^2, \quad i = 1, 2, 3,$$
$$h^{(1)} = e^{\eta_1} + e^{\eta_2} + e^{\eta_3}, \quad \eta_i = l_ix + m_it + \eta_i^{(0)}, \quad m_i = l_i^2, \quad i = 1, 2, 3,$$

we can obtain

$$\begin{aligned}
f^{(2)} &= e^{\xi_1+\eta_1+\theta_{14}} + e^{\xi_1+\eta_2+\theta_{15}} + e^{\xi_1+\eta_3+\theta_{16}} + e^{\xi_2+\eta_1+\theta_{24}} + e^{\xi_2+\eta_2+\theta_{25}} + e^{\xi_2+\eta_3+\theta_{26}} + e^{\xi_3+\eta_1+\theta_{34}} + e^{\xi_3+\eta_2+\theta_{35}} + e^{\xi_3+\eta_3+\theta_{36}},\\
g^{(3)} &= e^{\xi_1+\xi_2+\eta_2+\theta_{12}+\theta_{15}+\theta_{25}} + e^{\xi_2+\xi_3+\eta_1+\theta_{23}+\theta_{24}+\theta_{34}} + e^{\xi_2+\xi_3+\eta_2+\theta_{23}+\theta_{25}+\theta_{35}} + e^{\xi_1+\xi_2+\eta_1+\theta_{12}+\theta_{14}+\theta_{24}}\\
&\quad + e^{\xi_2+\xi_3+\eta_3+\theta_{23}+\theta_{26}+\theta_{36}} + e^{\xi_1+\xi_3+\eta_3+\theta_{13}+\theta_{16}+\theta_{36}} + e^{\xi_1+\xi_3+\eta_2+\theta_{13}+\theta_{35}+\theta_{15}} + e^{\xi_1+\xi_2+\eta_3+\theta_{12}+\theta_{16}+\theta_{26}}\\
&\quad + e^{\xi_1+\xi_3+\eta_1+\theta_{13}+\theta_{14}+\theta_{34}},\\
h^{(3)} &= e^{\xi_1+\eta_1+\eta_3+\theta_{14}+\theta_{16}+\theta_{46}} + e^{\xi_2+\eta_1+\eta_3+\theta_{24}+\theta_{26}+\theta_{46}} + e^{\xi_1+\eta_1+\eta_2+\theta_{56}+\theta_{15}+\theta_{45}} + e^{\xi_3+\eta_2+\eta_3+\theta_{35}+\theta_{36}+\theta_{56}}\\
&\quad + e^{\xi_1+\eta_2+\eta_3+\theta_{15}+\theta_{16}+\theta_{56}} + e^{\xi_2+\eta_1+\eta_2+\theta_{24}+\theta_{25}+\theta_{45}} + e^{\xi_3+\eta_1+\eta_2+\theta_{34}+\theta_{35}+\theta_{45}} + e^{\xi_3+\eta_1+\eta_3+\theta_{34}+\theta_{36}+\theta_{46}}\\
&\quad + e^{\xi_2+\eta_2+\eta_3+\theta_{25}+\theta_{26}+\theta_{56}},\\
f^{(4)} &= e^{\xi_2+\xi_3+\eta_1+\eta_3+\theta_{23}+\theta_{46}+\theta_{36}+\theta_{24}+\theta_{26}+\theta_{34}} + e^{\xi_2+\xi_3+\eta_2+\eta_3+\theta_{23}+\theta_{25}+\theta_{26}+\theta_{35}+\theta_{36}+\theta_{56}}\\
&\quad + e^{\xi_1+\xi_3+\eta_2+\eta_3+\theta_{13}+\theta_{15}+\theta_{16}+\theta_{35}+\theta_{36}+\theta_{56}} + e^{\xi_1+\xi_2+\eta_2+\eta_3+\theta_{12}+\theta_{15}+\theta_{16}+\theta_{25}+\theta_{26}+\theta_{56}}\\
&\quad + e^{\xi_2+\xi_3+\eta_1+\eta_2+\theta_{23}+\theta_{24}+\theta_{25}+\theta_{34}+\theta_{35}+\theta_{45}} + e^{\xi_1+\xi_3+\eta_1+\eta_2+\theta_{13}+\theta_{14}+\theta_{15}+\theta_{34}+\theta_{35}+\theta_{45}}\\
&\quad + e^{\xi_1+\xi_2+\eta_1+\eta_2+\theta_{12}+\theta_{14}+\theta_{15}+\theta_{24}+\theta_{25}+\theta_{45}} + e^{\xi_1+\xi_3+\eta_1+\eta_3+\theta_{34}+\theta_{13}+\theta_{14}+\theta_{16}+\theta_{36}+\theta_{46}}\\
&\quad + e^{\xi_1+\xi_2+\eta_1+\eta_3+\theta_{12}+\theta_{14}+\theta_{16}+\theta_{24}+\theta_{26}+\theta_{46}},\\
g^{(5)} &= e^{\xi_1+\xi_2+\xi_3+\eta_1+\eta_2+\theta_{12}+\theta_{13}+\theta_{23}}\cdot e^{\theta_{45}+\theta_{24}+\theta_{25}+\theta_{34}+\theta_{35}+\theta_{14}+\theta_{15}}\\
&\quad + e^{\xi_1+\xi_2+\xi_3+\eta_1+\eta_3+\theta_{12}+\theta_{23}+\theta_{13}}\cdot e^{\theta_{46}+\theta_{24}+\theta_{26}+\theta_{34}+\theta_{36}+\theta_{14}+\theta_{16}}\\
&\quad + e^{\xi_1+\xi_2+\xi_3+\eta_2+\eta_3+\theta_{12}+\theta_{13}+\theta_{23}}\cdot e^{\theta_{56}+\theta_{15}+\theta_{25}+\theta_{16}+\theta_{26}+\theta_{35}+\theta_{36}},\\
h^{(5)} &= e^{\xi_1+\xi_2+\eta_1+\eta_2+\eta_3+\theta_{12}+\theta_{45}+\theta_{46}+\theta_{56}}\cdot e^{\theta_{14}+\theta_{15}+\theta_{16}+\theta_{24}+\theta_{25}+\theta_{26}}\\
&\quad + e^{\xi_1+\xi_3+\eta_1+\eta_2+\eta_3+\theta_{13}+\theta_{45}+\theta_{46}+\theta_{56}}\cdot e^{\theta_{14}+\theta_{15}+\theta_{16}+\theta_{34}+\theta_{35}+\theta_{36}}\\
&\quad + e^{\xi_2+\xi_3+\eta_1+\eta_2+\eta_3+\theta_{23}+\theta_{45}+\theta_{46}+\theta_{56}}\cdot e^{\theta_{24}+\theta_{25}+\theta_{26}+\theta_{34}+\theta_{35}+\theta_{36}},\\
f^{(6)} &= e^{\xi_1+\xi_2+\xi_3+\eta_1+\eta_2+\eta_3+\theta_{12}+\theta_{13}+\theta_{23}+\theta_{45}+\theta_{46}+\theta_{56}}\cdot e^{\theta_{14}+\theta_{15}+\theta_{16}+\theta_{24}+\theta_{25}+\theta_{26}+\theta_{34}+\theta_{35}+\theta_{36}},
\end{aligned}$$

where $e^{\theta_{j,\rho}} = (k_j-k_\rho)^2$, $e^{\theta_{j,3+\rho}} = \frac{1}{(k_j+l_\rho)^2}$, $(j, \rho = 1, 2, 3)$, $e^{\theta_{n+j,3+\rho}} = (l_j-l_\rho)^2, (j < \rho = 2, 3)$. Taking

$f^{(2j)} = g^{(2j-1)} = h^{(2j-1)} = 0$, $j = 4, 5, \ldots$, we can get the three-soliton solution for the Levi equations from Eq. (2).

Generally, we have

$$f = \sum_{\mu=0,1} A_1(\mu) \exp\left(\sum_{j=1}^{2n} \mu_j \xi_j + \sum_{1\le j<\rho}^{2n} \mu_j \mu_\rho \theta jp\right), \tag{20}$$

$$g = \sum_{\mu=0,1} A_2(\mu) \exp\left(\sum_{j=1}^{2n} \mu_j \xi_j + \sum_{1\le j<\rho}^{2n} \mu_j \mu_\rho \theta jp\right), \tag{21}$$

where

$$\xi_j = k_j x + w_j t + \xi_j^{(0)}, \quad \eta_j = l_j x + m_j t + \eta_j^{(0)}, \quad (j = 1, 2, \ldots, n), \tag{22}$$

$$w_j = -k_j^2, \quad m_j = l_j^2, \xi_{n+j} = \eta_j,$$

$$e^{\theta_{j,n+\rho}} = \frac{1}{(k_j + l_\rho)^2}, \ (i, j = 1, 2, \ldots, n), \tag{23}$$

$$e^{\theta_{j,\rho}} = (k_j - k_\rho)^2, \quad e^{\theta_{n+j,n+\rho}} = (l_j - l_\rho)^2, \quad (j < \rho = 2, 3, \ldots, n), \tag{24}$$

$A_1(\mu)$ and $A_2(\mu)$ take over all possible combinations of $\mu_j = 0, 1 (j = 1, 2, \cdots, 2n)$ and satisfy the following condition

$$\sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \mu_{n+j}, \quad \sum_{j=1}^{n} \mu_j = 1 + \sum_{j=1}^{n} \mu_{n+j}, \tag{25}$$

respectively.

In the following, we use the abbreviated notion of Freeman and Nimmo[6] for the Wronskian and its derivatives. For convenience of proof, we first give the following lemmas

Lemma 1.

$$|Q, a, b||Q, c, d| - |Q, a, c||Q, b, d| + |Q, a, d||Q, b, c| = 0, \tag{26}$$

where Q is an $N \times (N-2)$ matrix, a, b, c and d represent N-dimensional column vectors.

Lemma 2.

$$\sum_{j=1}^{N} |\alpha_1, \ldots, \alpha_{j-1}, b\alpha_j, \alpha_{j+1}, \ldots, \alpha_N| = \sum_{j=1}^{N} b_j |\alpha_1, \alpha_2, \ldots, \alpha_N| = \sum_{j=1}^{N} b_j |A|, \tag{27}$$

where $\alpha_j \, (j = 1, 2, \ldots, N)$ are N column vectors, $b = (b_1, b_2, \ldots, b_N)^T$ and $b\alpha_j$ means $(b_1\alpha_{1j}, b_2\alpha_{2j}, \ldots, b_N\alpha_{Nj})^T$.

Applying the Wronskian technique, we can obtain *Theorem* 1. The Levi equations (1) have the double Wronskian solutions

$$f = |\, \widehat{N}; \widehat{M}\,|, \quad g = 2\,|\, \widehat{N+1}; \widehat{M-1}\,|, \quad h = 2\,|\, \widehat{N-1}; \widehat{M+1}\,|, \tag{28}$$

where the entries $\phi_j, \psi_j (j = 1, 2, \cdots, N)$ enjoy the following conditions

$$\phi_{j,x} = \lambda_j \phi_j, \quad \psi_{j,x} = -\lambda_j \psi_j, \tag{29}$$

$$\phi_{j,t} = -2\phi_{j,xx}, \quad \psi_{j,t} = 2\psi_{j,xx}, \tag{30}$$

respectively.

Substituing $f, f_x, f_{xx}, f_t, g, g_x, g_{xx}, g_t$ into Eq. (3), we have

$$\begin{aligned}
&(D_t - D_x^2) f \cdot g = f_t g - f g_t - f_{xx} g + 2 f_x g_x - f g_{xx} \\
&= 2\,|\, \widehat{N+1}; \widehat{M-1}\,|\, \big(\,|\, \widehat{N-2}, N, N+1; \widehat{M}\,| \\
&\quad + 3\,|\, \widehat{N-1}, N+2; \widehat{M}\,| \\
&\quad - 2\,|\, \widehat{N-1}, N+1; \widehat{M-1}, M+1\,| \\
&\quad - 3\,|\, \widehat{N}; \widehat{M-2}, M, M+1\,| + |\, \widehat{N}; \widehat{M-1}, M-2\,|\,\big) \\
&\quad - |\, \widehat{N}; \widehat{M}\,|\, \big(6\,|\, \widehat{N-1}, N+1, , N+2; \widehat{M-1}\,| \\
&\quad - 2\,|\, \widehat{N}, N+3; \widehat{M-1}\,| + 4\,|\, \widehat{N}, N+2; \widehat{M-2}, M\,| \\
&\quad + 2\,|\, \widehat{N+1}; \widehat{M-3}, M-1, M\,| \\
&\quad + 6\,|\, \widehat{N+1}; \widehat{M-2}, M+1\,|\,\big), \\
&\quad + 4\big(\,|\, \widehat{N}, N+2; \widehat{M-1}\,| + |\, \widehat{N+1}; \widehat{M-2}, M\,|\,\big) \\
&\quad \cdot \big(|\widehat{N-1}, N+1; \widehat{M}| + |\widehat{N}; \widehat{M-1}, M+1|\big).
\end{aligned} \tag{31}$$

Noting

$$\begin{aligned}
&\left(\sum_{j=1}^{2N} \lambda_j\right)^2 |\, \widehat{N+1}; \widehat{M-1} \,||\, \widehat{N}; \widehat{M}\,| \\
&= \sum_{j=1}^{2N} \lambda_j\,|\, \widehat{N+1}; \widehat{M-1}\,| \sum_{j=1}^{2N} \lambda_j\,|\, \widehat{N}; \widehat{M}\,| \\
&= |\, \widehat{N+1}; \widehat{M-1}\,| \left(\sum_{j=1}^{2N} \lambda_j\right)^2 |\, \widehat{N}; \widehat{M}\,|\,.
\end{aligned} \tag{32}$$

That is,

$$\begin{aligned}
&|\, \widehat{N}; \widehat{M}\,|\, \big(\,|\, \widehat{N-1}, N+1, N+2; \widehat{M-1}\,| \\
&\quad + |\, \widehat{N}, N+3; \widehat{M-1}\,| - 2\,|\, \widehat{N}, N+2; \widehat{M-2}, M\,| \\
&\quad + |\, \widehat{N+1}; \widehat{M-3}, M-1, M\,| \\
&\quad + |\, \widehat{N+1}; \widehat{M-2}, M+1\,|\,\big) \\
&= \big(\,|\, \widehat{N}, N+2; \widehat{M-1}\,| - |\, \widehat{N+1}; \widehat{M-2}, M\,|\,\big) \\
&\quad \cdot \big(\,|\, \widehat{N-1}, N+1; \widehat{M}\,| - |\, \widehat{N}; \widehat{M-1}, M+1\,|\,\big) \\
&= |\, \widehat{N+1}; \widehat{M-1}\,|\, \big(\,|\, \widehat{N-2}, N, N+1; \widehat{M}\,| \\
&\quad + \widehat{N-1}, N+2; \widehat{M}\,| - 2\,|\, \widehat{N-1} N+1; \\
&\quad \widehat{M-1}, M+1\,| + |\, \widehat{N}; \widehat{M-2}, M, M+1\,| \\
&\quad + |\, \widehat{N}; \widehat{M-1}, M+2\,|\big).
\end{aligned} \tag{33}$$

Utilizing the identity (33), the right of Eq. (31) is reduced as

$$\begin{aligned}
&-4|\widehat{N}, N+2; \widehat{M-1}||\widehat{N-1}, N+1; \widehat{M}| \\
&-4|\widehat{N+1}; \widehat{M-2}, M||\widehat{N}; \widehat{M-1}, M+1| \\
&+4|\widehat{N}; \widehat{M}||\widehat{N-1}, N+1, N+2; \widehat{M-1}| \\
&+4|\widehat{N}; \widehat{M}||\widehat{N+1}; \widehat{M-2}, M+1|
\end{aligned}$$

$$+4|\widehat{N+1};\widehat{M-1}||\widehat{N-1},N+2;\widehat{M}|$$
$$+4|\widehat{N+1};\widehat{M-1}||\widehat{M-2},M,M+1|. \quad (34)$$

Using *Lemma* 1, (34) is equal to zero. So the proof of (3) is finished.
Similarly

$$\begin{aligned}
&D_x^2 f\cdot f-2gh=2f_{xx}f-2f_x^2-2gh\\
=&2\big[|\widehat{N-1};\widehat{M-1}|\big(|\widehat{N-1},N+2;\widehat{M}|\\
&+|\widehat{N-2},N,N+1;\widehat{M}|\\
&+2|\widehat{N-1},N+1;\widehat{M-1},M+1|\\
&+|\widehat{N};\widehat{M-1},M+2|+|\widehat{N};\widehat{M-2},M,M+1|\big)\\
&-\big(|\widehat{N-1},N+1;\widehat{M}|+|\widehat{N};\widehat{M-1},M+1|\big)^2\\
&-4|\widehat{N+1};\widehat{M-1}||\widehat{N-1};\widehat{M+1}|\big], \quad (35)
\end{aligned}$$

using the following identities

$$\begin{aligned}
&|\widehat{N-1},N+1;\widehat{M}|(|\widehat{N-1},N+1;\widehat{M}|\\
&-|\widehat{N};M-1,\widehat{M}|)\\
=&|\widehat{N};\widehat{M}|(|\widehat{N-2},N,N+1;\widehat{M}|+|\widehat{N-1},N+2;\widehat{M}|\\
&-|\widehat{N-1},N+1;\widehat{M-1},M+1|). \quad (36)\\
&|\widehat{N};\widehat{M-1},M+1|(|\widehat{N-1},N+1;\widehat{M}|\\
&-|\widehat{N};\widehat{M-1},M+1|)\\
=&|\widehat{N};\widehat{M}|\times(|\widehat{N-1},N+1;\widehat{M-1},M+1|\\
&-|\widehat{N};\widehat{M-2},M,M+1|\\
&-|\widehat{N};\widehat{M-1},M+2|). \quad (37)
\end{aligned}$$

We can have

$$\begin{aligned}
&2f_{xx}f-2f_x^2-2gh\\
=&8|\widehat{N};\widehat{M}||\widehat{N-1},N+1;\widehat{M-1},M+1|\\
&-8|\widehat{N-1},N+1;\widehat{M}||\widehat{N};\widehat{M-1},M+1|\\
&-8|\widehat{N+1};\widehat{M-1}||\widehat{N-1};\widehat{M+1}|, \quad (38)
\end{aligned}$$

according to *Lemma* 1, (38) is equal to zero.

In the same way,

$$\begin{aligned}
&D_tD_xf\cdot f-2D_xh\cdot g=2(f_{tx}f-f_tf_x-h_xg+hg_x)\\
=&4\big[|\widehat{N-1};\widehat{M-1}|(|\widehat{N-3},N-1,N,N+1;\widehat{M}|\\
&+|\widehat{N-2},N,N+1;\widehat{M-1},M+1|\\
&-|\widehat{N-1},N-3;\widehat{M}|-|\widehat{N-1},N+2;\widehat{M-1},M+1|\\
&-|\widehat{N-1},N+1;\widehat{M-2},M,M+1|\\
&-|\widehat{N};\widehat{M-3},M-1,M,M+1|\\
&+|\widehat{N-1},N+1;\widehat{M-1},M+2|\\
&+|\widehat{N};\widehat{M-1},M+3|)-(|\widehat{N-1},N+1;\widehat{M}|\\
&+|\widehat{N};\widehat{M-1},M+1|)(|\widehat{N-2},N,N+1;\widehat{M}|\\
&-|\widehat{N-1},N+2;\widehat{M}|-|\widehat{N};\widehat{M-2},M,M+1|\\
&+|\widehat{N};\widehat{M-1},M+2|)\\
&-2|\widehat{N+1};\widehat{M-1}|\big(|\widehat{N-2},N;\widehat{M+1}|\\
&+|\widehat{N-1};\widehat{M},M+2|\big)+2|\widehat{N-1};\widehat{M+1}|\\
&\cdot\big(|\widehat{N},N+2;\widehat{M-1}|+|\widehat{N+1};\widehat{M-2},M|\big)\big]. \quad (39)
\end{aligned}$$

Making use of the following identities which are similar to Eqs. (33),(36) and (37),

$$\begin{aligned}
&|\widehat{N};\widehat{M}|\big(|\widehat{N-3},N-1,N,N+1;\widehat{M}|\\
&-|\widehat{N-2},N,N+1;\widehat{M-1},M+1|\\
&-|\widehat{N-1},N+3;\widehat{M}|\\
&+|\widehat{N-1},N+2;\widehat{M-1},M+1|\\
&-|\widehat{N-1},N+1;\widehat{M-2},M,M+1|\\
&-|\widehat{N};\widehat{M-1},M+3|\\
&+|\widehat{N};\widehat{M-3},M-1,M,M+1|\\
&+|\widehat{N-1},N+1;\widehat{M-1},M+2|\big)\\
=&\big(|\widehat{N-1},N+1;\widehat{M}|-|\widehat{N};\widehat{M-1},M+1|\big)\\
&\cdot\big(|\widehat{N-2},N,N+1;\widehat{M}|-|\widehat{N-1},N+2;\widehat{M}|\\
&-|\widehat{N};\widehat{M-2},M,M+1|\\
&+|\widehat{N};\widehat{M-1},M+2|\big), \quad (40)\\
&\big(|\widehat{N-1},N+1;\widehat{M}|-|\widehat{N};\widehat{M-1},M+1|\big)\\
&\cdot\big(|\widehat{N};\widehat{M-2},M,M+1|-|\widehat{N};\widehat{M-1},M+2|\big)\\
=&|\widehat{N};\widehat{M}|\big(|\widehat{N-1},N+1;\widehat{M-2},M,M+1|\\
&-|\widehat{N};\widehat{M-3},M-1,M,M+1|\\
&-|\widehat{N-1},N+1;\widehat{M-1},M+2|\\
&+|\widehat{N};\widehat{M-1},M+3|\big). \quad (41)
\end{aligned}$$

From Lemma 1, we can find that Eq. (39) is equal to zero.
Thus we have completed the proof of Theorem 1.

The authors would like to thank the referees for their valuable suggestions and comments. The first author would like to express his sincere thanks to Professor Chen D Y for his invaluable guidance and help.

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