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Spatially Homogeneous Bianchi Type V Cosmological Model in the Scale-Covariant Theory of Gravitation

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We discuss spatially homogeneous and anisotropic Bianchi type-V spacetime filled with a perfect fluid in the framework of the scale-covariant theory of gravitation proposed by Canuto et al. By applying the law of variation for Hubble's parameter, exact solutions of the field equations are obtained, which correspond to the model of the universe having a big-bang type singularity at the initial time $t = 0$. The cosmological model, evolving from the initial singularity, expands with power-law expansion and gives essentially an empty space for a large time. The physical and dynamical properties of the model are also discussed.

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Several alternative theories of gravitation have been extensively studied by many workers. Canuto *et al.*[1,2] formulated the self-covariant theory of gravitation by associating the mathematical operation of scale transformation with the physics of using different dynamical systems to measure spacetime distances. They obtained the generalized Einstein's field equations invariant under scale transformations and studied several astrophysical tests. This theory is a viable alternative to general relativity which allows a natural interpretation of the possible variation of the gravitational constant G (Wesson[3] and Will[4]). Beesham[5] has discussed power asymptotic singularities in the scale-covariant theory of gravitation with special attention to the Friedmann model and the Kasner model, and has generalized the corresponding general relativistic results. In this theory Einstein's field equations are valid in gravitational units whereas physical quantities are measured in atomic units. The metric tensors in the two systems of units are related by the conformal transformation

$$\bar{g}_{\mu\nu} = \phi^2 g_{\mu\nu}, \tag{1}$$

where the gauge function ϕ is the function of coordinates. A bar denotes gravitational units and unbarred denotes atomic units. Using Eq. (1), Canuto *et al.*[1] transformed the usual Einstein's field equations into

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + f_{\mu\nu}(\phi) = -8\pi G(\phi)T_{\mu\nu} + \Lambda(\phi)g_{\mu\nu}, \tag{2}$$

where

$$\phi^2 f_{;\mu\nu} = 2\phi\phi_{;\mu\nu} - 4\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}(\phi\phi^{;\mu}_{;\mu} - \phi^{,\mu}\phi_{,\mu}), \tag{3}$$

where a semicolon denotes covariant differentiation and comma denotes ordinary differentiation.

For a perfect fluid the energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \tag{4}$$

where ρ is the energy-density of matter, p the pressure and u^μ is the four-velocity vector of the fluid satisfying $u^\mu u_\mu = 1$.

Recently, Reddy *et al.*[6] have presented an LRS Bianchi type I cosmological model with a negative constant deceleration parameter in the scale-covariant theory of gravitation. It is certainly of interest to study cosmologies with a richer structure both geometrically and physically. Bianchi type V models are of particular interest since they are sufficiently complex as the Einstein tensor has off-diagonal terms while at the same time they are simple generalizations of negative curvature FRW models. Such a type of model has a significant role in the description of the universe at the early stages of its evolution. Zhu and Zhu[7] have presented a solution of the Einstein field equations in a Bianchi V universe with coherent radiation and have shown that with the lapse of time, the model approaches the state of isotropic expansion.

In this Letter, we discuss Einstein's field equations in the scale-covariant theory of gravitation for an anisotropic Bianchi type V model filled with perfect fluid. By applying the variation law for the generalized mean Hubble parameter we derive two types of solutions for the average scale factor, one is of power-law type and the other is of an exponential form. Using the power-law from the average scale factor, we obtain exact solutions of the field equations. The set of solutions corresponds to a model of the universe which evolves from an initial big-bang singularity at $t = 0$ and expands with power-law expansion. The deceleration parameter for the cosmological model is positive. The model essentially gives an empty space for a large time. We also discuss some physical properties of the model.

We consider the spatially homogeneous and anisotropic Bianchi type V metric in the form

$$ds^2 = dt^2 - A^2(t)dx^2 - e^{2mx}\left[(B^2(t)dy^2 + C^2(t)dz^2\right], \tag{5}$$

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where A, B and C are functions of the cosmic time t and m is a constant.

In comoving coordinates, the field equations (2)–(4) for the metric (5) can be written as

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}}{B}\frac{\dot{C}}{C}-\frac{m^2}{A^2}-2\frac{\dot{A}}{A}\frac{\dot{\phi}}{\phi}+\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)+\frac{\ddot{\phi}}{\phi}-\frac{\dot{\phi}^2}{\phi^2}=-8\pi Gp, \quad (6)$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}}{A}\frac{\dot{C}}{C}-\frac{m^2}{A^2}-2\frac{\dot{B}}{B}\frac{\dot{\phi}}{\phi}+\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)+\frac{\ddot{\phi}}{\phi}-\frac{\dot{\phi}^2}{\phi^2}=-8\pi Gp, \quad (7)$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}}{A}\frac{\dot{B}}{B}-\frac{m^2}{A^2}-2\frac{\dot{C}}{C}\frac{\dot{\phi}}{\phi}+\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)+\frac{\ddot{\phi}}{\phi}-\frac{\dot{\phi}^2}{\phi^2}=-8\pi Gp, \quad (8)$$

$$\frac{\dot{A}}{A}\frac{\dot{B}}{B}+\frac{\dot{A}}{A}\frac{\dot{C}}{C}+\frac{\dot{B}}{B}\frac{\dot{C}}{C}-\frac{3m^2}{A^2}+\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)-\frac{\ddot{\phi}}{\phi}+3\frac{\dot{\phi}^2}{\phi^2}=8\pi G\rho, \quad (9)$$

$$2\frac{\dot{A}}{A}-\frac{\dot{B}}{B}-\frac{\dot{C}}{C}=0. \quad (10)$$

The energy conservation equation which is a consequence of the field equations (1) and (2), in this theory, is given by (Canuto *et al.*[1])

$$\dot{\rho}+(\rho+p)u^{\mu}_{;\mu}=-\rho\left(\frac{\dot{G}}{G}+\frac{\dot{\phi}}{\phi}\right)-3p\frac{\dot{\phi}}{\phi}. \quad (11)$$

An overdot denotes differentiation with respect to t. For the metric (5), this takes the form

$$\dot{\rho}+(\rho+p)\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)+\rho\left(\frac{\dot{G}}{G}+\frac{\dot{\phi}}{\phi}\right)+3p\frac{\dot{\phi}}{\phi}=0. \quad (12)$$

Now we define certain physical and kinematical parameters. For the metric (5), the expansion scalar θ and the shear scalar σ have the usual definitions and are given by

$$\theta=u^{\mu}_{;\mu}=\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right), \quad (13)$$

$$\sigma^2=\frac{1}{2}\sigma_{\mu\upsilon}\sigma^{\mu\upsilon}=\frac{1}{2}\left[\left(\frac{\dot{A}}{A}\right)^2+\left(\frac{\dot{B}}{B}\right)^2+\left(\frac{\dot{C}}{C}\right)^2\right]-\frac{\theta^2}{6}. \quad (14)$$

The generalized mean Hubble's parameter H is defined by

$$H=\frac{1}{3}(H_1+H_2+H_3), \quad (15)$$

where $H_1=\dot{A}/A$, $H_2=\dot{B}/B$ and $H_3=\dot{C}/C$ are directional Hubble's parameters in the directions of x, y and z, respectively. The volume scalar V is defined as

$$V=R^3=ABC, \quad (16)$$

where R is the average scale factor. From Eqs. (13) and (14), we obtain

$$H=\frac{1}{3}\frac{\dot{V}}{V}=\frac{\dot{R}}{R}=\frac{1}{3}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right). \quad (17)$$

The deceleration parameter q, in terms of the average scale factor R, is defined by

$$q=-\frac{\ddot{R}R}{\dot{R}^2}. \quad (18)$$

From Eqs. (6)–(9), we obtain the energy-density and the pressure in terms of the above kinematical parameters as

$$8\pi G\rho=3H^2-\sigma^2-\frac{3m^2}{A^2}-\frac{\ddot{\phi}}{\phi}+3\left(\frac{\dot{\phi}}{\phi}\right)^2+3H\frac{\dot{\phi}}{\phi}, \quad (19)$$

$$8\pi Gp=H^2(2q-1)-\sigma^2+\frac{m^2}{A^2}-\frac{\ddot{\phi}}{\phi}+\left(\frac{\dot{\phi}}{\phi}\right)^2-H\frac{\dot{\phi}}{\phi}. \quad (20)$$

Integrating Eq. (10) and absorbing the integration constant in the function B or C, we obtain

$$A^2=BC. \quad (21)$$

We follow the technique used by Saha and Rikhvitsky,[8] Singh and Chaubey,[9] Singh *et al.*[10], and Shri Ram *et al.*[11,12] to solve the field equations (6)–(12) in quadrature forms. Subtracting Eq. (6) from Eq. (7), Eq. (7) from Eq. (6) and Eq. (6) from Eq. (8) respectively and integrating the results, we obtain

$$\frac{B}{A}=d_1\exp\left(k_1\int\frac{dt}{R^3\phi^2}\right), \quad (22)$$

$$\frac{C}{B}=d_2\exp\left(k_2\int\frac{dt}{R^3\phi^2}\right), \quad (23)$$

$$\frac{C}{A}=d_3\exp\left(k_3\int\frac{dt}{R^3\phi^2}\right) \quad (24)$$

where d_1, d_2, d_3 and k_1, k_2, k_3 are constants of integration. Going through further straightforward mathematical calculations, the metric functions A, B and C can be obtained explicitly as

$$A(t)=l_1R\exp\left(X_1\int\frac{dt}{R^3\phi^2}\right), \quad (25)$$

$$B(t)=l_2R\exp\left(X_2\int\frac{dt}{R^3\phi^2}\right), \quad (26)$$

$$C(t)=l_3R\exp\left(X_3\int\frac{dt}{R^3\phi^2}\right), \quad (27)$$

with $l_1=\sqrt[3]{(d_1{}^2d_2)^{-1}}$, $l_2=\sqrt[3]{d_1d_2{}^{-1}}$, $l_3=\sqrt[3]{d_1d_2{}^2}$, and $X_1=-\frac{1}{3}(2k_1+k_2)$, $X_2=-\frac{1}{3}(k_2-k_1)$, $X_3=$

$\frac{1}{3}(k_1+2k_2)$. It can be seen that the constants X_1, X_2, X_3 and l_1, l_2, l_3 satisfy the following relations:

$$X_1+X_2+X_3=0, \quad \text{and} \quad l_1l_2l_3=1. \tag{28}$$

Substituting Eq. (21) into Eqs. (25)–(27), we find

$$X_1=0,\ X_2=-X_3,\ \text{and}\ l_1=1,\ l_2=l_3^{-1}=b, \tag{29}$$

where X and b are constants. Substituting Eq. (29) into Eqs. (25)–(27), the quadrature solutions for A, B and C can be written as

$$A(t)=R, \tag{30}$$

$$B(t)=bR\exp\left(X\int\frac{dt}{R^3\phi^2}\right), \tag{31}$$

$$C(t)=b^{-1}R\exp\left(-X\int\frac{dt}{R^3\phi^2}\right). \tag{32}$$

It is clear that we can determine A, B and C as functions of cosmic time t if the average scale factor R and the gauge function ϕ are known.

Now we will find the scale factor R by applying a special law of variation for Hubble's parameter which yields constant values of deceleration parameter. We assume that Hubble's parameter H is expressed in terms of the average scale factor R as

$$H=lR^{-n}, \tag{33}$$

where $l>0$ and $n\geq 0$ are constants. Initially this type of relation was considered by Berman[13] and Berman and Gomide.[14] Later on Singh and Kumar[15] and Kumar and Singh[16] have applied the concept of the constant deceleration parameter for solving Einstein's field equations in Bianchi type I and II cosmological metrics. Recently Singh *et al.*[10] and Shri Ram *et al.*[11,12] have extended this approach to Bianchi type V models in general relativity and some scalar-tensor theories of gravitation.

From Eqs. (17) and (33), we obtain

$$\dot{R}=lR^{-n+1}, \tag{34}$$

$$\ddot{R}=-l^2(n-1)R^{-2n+1}. \tag{35}$$

Substituting Eqs. (34) and (35) into Eq. (18), we obtain

$$q=n-1. \tag{36}$$

Thus, the deceleration parameter q is constant. It is well known that for the positive and negative values of q, the model decelerates and accelerates respectively.

Integration of Eq. (34) yields $R=(nlt+c_1)^{1/n}$, $n\neq 0$ and $R=c_2\exp(lt)$, n=0, where c_1 and c_2 are integration constants. Without loss of any generality we can take $c_1=0$ and $c_2=1$. Then, we have

$$R=(nlt)^{1/n}, \quad n\neq 0, \tag{37}$$

and

$$R=\exp(lt), \quad n=0. \tag{38}$$

In Eq. (37) the average scale factor R is of power-law form whereas in Eq. (38) it is of exponential form. We now use the power-law solution for R to obtain integrals in Eqs. (31) and (32).

We now obtain the solutions of field equations for the value of R given in Eq. (37). In Eqs. (31) and (32), the gauge function ϕ appears under the integration sign. The integrals in Eqs. (31) and (32) can be evaluated if the gauge function ϕ is known as a function of time. We assume the functional form of ϕ as

$$\phi=\beta R^{\alpha}=\beta(nlt)^{\alpha/n}, \tag{39}$$

where α and β are constants. Substituting Eqs. (37) and (39) into Eqs. (31) and (32) and integrating, we obtain the solutions of metric functions as

$$A=(nlt)^{1/n}, \tag{40}$$

$$B(t)=b(nlt)^{1/n}\exp\left[\frac{X}{l\beta^2(n-3-2\alpha)}\cdot(nlt)^{(n-3-2\alpha)/n}\right], \tag{41}$$

$$C(t)=b^{-1}(nlt)^{1/n}\exp\left[\frac{-X}{l\beta^2(n-3-2\alpha)}\cdot(nlt)^{(n-3-2\alpha)/n}\right], \tag{42}$$

provided $n\neq 3+2\alpha$.

The expansion scalar and shear scalar are obtained as

$$\theta=3/(nt), \tag{43}$$

$$\sigma^2=\frac{X^2}{\beta^4}(nlt)^{-(4\alpha+6)/n}. \tag{44}$$

The directional Hubble's parameters and the generalized Hubble's parameter are given as follows:

$$H_1=1/nt, \tag{45}$$

$$H_2=1/nt+\frac{X}{\beta^2}(nlt)^{-(2\alpha+3)/n}, \tag{46}$$

$$H_3=1/nt-\frac{X}{\beta^2}(nlt)^{-(2\alpha+3)/n}, \tag{47}$$

$$H=1/nt. \tag{48}$$

The expressions for energy density and pressure, as obtained from Eqs. (19) and (20), are given by

$$8\pi G\rho=(2\alpha^2+\alpha n+3\alpha+3)(nt)^{-2}-\frac{X^2}{\beta^4}(nlt)^{-(4\alpha+6)/n}-3m^2(nlt)^{-2/n}, \tag{49}$$

$$8\pi Gp=(2n+\alpha n-\alpha-3)(nt)^{-2}-\frac{X^2}{\beta^4}(nlt)^{-(4\alpha+6)/n}+m^2(nlt)^{-2/n}. \tag{50}$$

In Eqs. (49) and (50), the gravitational term $G(t)$ is arbitrary.

It can be seen that the conservation equation (12) is satisfied with arbitrary G if

$$\alpha = n - 3. \tag{51}$$

Hence, the metric functions of the cosmological model can be written explicitly as

$$A = (nlt)^{1/n}, \tag{52}$$

$$B(t) = b(nlt)^{\frac{1}{n}} \exp\left[\frac{-X}{l\beta^2(n-3)}(nlt)^{-\frac{n-3}{n}}\right], \tag{53}$$

$$C(t) = b^{-1}(nlt)^{\frac{1}{n}} \exp\left[\frac{X}{l\beta^2(n-3)}(nlt)^{-\frac{n-3}{n}}\right]. \tag{54}$$

To find the energy-density and pressure in the forms independent of G, we recall that in most of the variable G cosmologies, G is a decreasing function of time. The possibility of an increasing G has also been discussed by Levit.[17] Beesham[18] has discussed the possibility of a creation field with $G \propto t^{\lambda}$, λ = constant. Sistero[19] has presented exact solutions for zero pressure Robertson–Walker cosmological models with $G \propto R^{\lambda}$. Here we obtain a physically realistic model of the universe by assuming a time-dependent G of the form

$$G = kt, \tag{55}$$

where k is a proportionality constant. Using Eqs. (51) and (52) into Eqs. (49) and (50), we obtain

$$8\pi\rho = \frac{3n^2 - 12n + 12}{n^2t^3} - \frac{X^2}{\beta^4}\frac{(nlt)^{-(4n-6)/n}}{t} - 3m^2\frac{(nlt)^{-2/n}}{t} \tag{56}$$

$$8\pi p = \frac{n^2 - 2n}{n^2t^3} - \frac{X^2}{\beta^4}\frac{(nlt)^{-(4n-6)/n}}{t} + m^2\frac{(nlt)^{-2/n}}{t}. \tag{57}$$

We now discuss the physical and kinematical behaviors of the cosmological model represented by this set of exact solutions. We observe that the spatial volume and the three scale factors are zero at $t = 0$. At this epoch the energy-density and pressure assume infinite values for $1 \le n \le 3$. The expansion scalar is infinite at $t = 0$. The shear scalar is also infinite at $t = 0$ for $n \ge 3/2$. The deceleration parameter is positive for $n > 1$. Therefore this model describes a continuously expanding, shearing and decelerating universe with a big-bang start at $t = 0$. All the kinematical parameters θ, σ, H_1, H_2, H_3 and H are well behaved and are decreasing functions of time for $t > 0$. With an increase in time, the rate of volume expansion slows down and in the limit of large time, i.e., $t \to \infty$, the energy-density, pressure, expansion scalar and shear become zero for $2 \le n \le 3$. Therefore for a physically realistic model of the universe satisfying the conditions at $t = 0$ and $t \to \infty$, we must have $2 \le n \le 3$. This model essentially gives an empty space for a large time. We also find that σ/θ tends to zero as $t \to \infty$, which indicates that the model approaches isotropy for large values of t. The gravitational $G(t)$ is zero initially and gradually increases and tends to infinity at late times.

We have also attempted to obtain exact solutions of the field equations for the exponential form of R given in Eq. (38) with $\phi = \beta R^{\alpha}$ and $G = kt$. However, the corresponding model of the universe is found physically unrealistic as in that model the energy-density and pressure do not tend to zero or constant as $t \to \infty$.

In summary, we have considered the field equations of the scale-covariant theory of gravitation proposed by Canuto *et al.* for an anisotropic and spatially homogeneous Bianchi type V spacetime. We have obtained exact solutions of the field equations by applying the law of variation for Hubble's parameter. The corresponding cosmological model represents an expanding and decelerating universe having an initial big-bang singularity at $t = 0$. We have also discussed the physical properties of the cosmological model which essentially gives an empty space for a large time.

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