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Cosmological Models with Time Dependent G and Λ Coupling Scalars

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A cosmological model in which the universe has its critical density and gravitational constants generalized as coupling scalars in Einstein's theory is considered. A general method of solving the field equations is given. An exact solution for matter distribution in cosmological models satisfying $G = G_0(R/R_0)^n$ is presented. Corresponding physical interpretations of the cosmological solutions are also discussed.

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A theory of gravitation using G and Λ as non-constant coupling scalars has been used by Beesham[1] and Abdel-Rahman.[2] The motivation was to include a G-coupling 'constant' of gravity as pioneered by Dirac in 1937.[3] It is a straightforward generalization of Einstein's equations

$$S^{ab} \equiv R^{ab} - \frac{1}{2}Rg^{ab} = -8\pi GT^{ab} - \Lambda g^{ab}, \quad (1)$$

where S^{ab} is Einstein's tensor, T^{ab} the matter energy-momentum tensor, g^{ab} the metric tensor, G and Λ are coupling scalars. If we assume the equality of gravitational and inertial mass and the gravitational time dilation as in Einstein theory, we must require that the equation of motion of particles and photons does not contain G and Λ but only g^{ab}. Thus the interchange of energy between matter and gravitation is given by the usual conservation laws

$$T^{ab}_{;b} = 0, \quad (2a)$$

and the Bianchi identity

$$S^{ab}_{;a} = 0. \quad (2b)$$

Therefore, there is a restriction in the role of the scalars G and Λ to the effects on the field equation (1) and once g^{ab} is determined, as in Einstein's theory, the gravitational phenomena are described in the same way.

Taking into account the Bianchi identity (2b) and (2a)(in the presence of a non-constant cosmological term and gravitational constant), the covariant divergence of Eq. (1) gives

$$8\pi G_{;b}T^{ab} + \Lambda_{;b}g^{ab} = 0. \quad (3)$$

If G and Λ depend on the cosmological time t, the covariant conservative (3) is equivalent to ($a, b, c = 1, 2, 3$ denote the space indices)

$$(GT^a_i)_{;a} = 0, \quad (4)$$

$$(GT^a_0)_{;a} = -\Lambda_{;0}. \quad (5)$$

Cosmological models based on these equations may be useful to study the early universe (singular or not) and their relations with particle fields as they allow the possibility of investigating different cases for G, as in Dirac's cosmology, for example, or to solve the same cosmological difficulties as in Ozer and Taha.[4] In any case the strongest constraints are the presently observed G_0 value and the observational limits of Λ_0. Sistero[5] found an exact solution for zero pressure models satisfying $G = G_0(R/R_0)^n$. We also previously found an exact solution for Zeldovich fluid models satisfying $G = G_0(R/R_0)^n$.[6] Barrow[7] investigated the Newtonian form of gravity theories with varying G and exact solutions and all asymptotic cosmological behavior is found for a universe with $G \propto t^{-n}$.

In this Letter, we present an exact solution for matter distribution in cosmological models satisfying $G = G_0(R/R_0)^n$.

We consider the spacetime described by the isotropic homogeneous Robertson-Walker metric

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2\right], \quad (6)$$

where k is the curvature index which can take up the values $(-1, 0, +1)$ corresponding to the open, flat and closed universe respectively and $R(t)$ is the scale factor. The universe is assumed to be filled with distribution of matter represented by the energy-momentum tensor of a perfect fluid

$$T^{ab} = -pg^{ab} + (p + \rho)u^au^b, \quad (7)$$

where p and ρ are the pressure and energy density of the cosmic matter respectively and u^a is the four-velocity vector such that $u_vu^v = 1$.

Using co-moving coordinates

$$u^a = (1, 0, 0, 0), \quad (8)$$

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in Eq. (7) and with the line element (6), Einstein's field equation (1) yields two independent equations

$$3\ddot{R} = -4\pi GR\Big(3p + \rho - \frac{\Lambda}{4\pi G}\Big), \tag{9}$$

$$3\dot{R}^2 = 8\pi GR^2\Big(\rho + \frac{\Lambda}{8\pi G}\Big) - 3k. \tag{10}$$

In uniform cosmology $G = G(t)$ and $\Lambda = \Lambda(t)$, so that the conservation equation (3) is given by

$$\dot{\Lambda} = -8\pi\dot{G}\rho, \tag{11}$$

where the overdot denotes differentiation with respect to t. Equations (9)–(11) are the fundamental equations. Equations (9) and (10) may be written as

$$8\pi Gp = -2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2} + \Lambda, \tag{12}$$

$$8\pi G\rho = \frac{3(k + \dot{R}^2)}{R^2} - \Lambda. \tag{13}$$

Eliminating Λ between Eq. (12) and Eq. (13), $\ddot{R}$ with the derivative of Eq. (13) and using Eq. (11), it is found that

$$\frac{d}{dt}(\rho R^3) + p\frac{d(R^3)}{dt} = 0. \tag{14}$$

It is clear that Eqs. (10) and (11) are formally identical to those of the usual cosmology with G and Λ constants as must be, since the left-hand side of Eq. (1) depends only on the metric components (6) and G and Λ enter algebraically on the right-hand side of Eq. (1). Also, Eq. (14) is identical to that of the usual cosmology despite the fact that it comes from a differential form of Eq. (13) and from Eq. (12), both involving the time dependent scalars G and Λ; however, their derivatives eliminate with Eq. (11), thus leading to Eq. (14).

Equations (11), (13) and (14) will be used in the following as fundamental since they are independent. The cosmological problem modeled by these equations leaves two degrees of freedom; it may be determined by a physical assumption $p = p(\rho)$, i.e. the equation of state and form an additional explicit adoption on R, ρ, p, G or Λ in terms of t or R which itself depends on t. Once the problem is determined the integration constants are characterized by the observable parameters

$$H_0 = \frac{\dot{R}_0}{R_0}, \tag{15}$$

$$\sigma_0 = \frac{4\pi}{3}\frac{G_0\rho_0}{H_0^2}, \tag{16}$$

$$q_0 = -\frac{\ddot{R}_0}{R_0}H_0^2, \tag{17}$$

$$\varepsilon_0 = \frac{p_0}{\rho_0}, \tag{18}$$

which must satisfy Einstein's equations at present cosmic time t_0:

$$\Lambda_0 = 3H_0^2[(1 + 3\varepsilon_0)\sigma_0 - q_0], \tag{19}$$

$$\frac{k}{R_0^2} = H_0^2\big[3(1 + \varepsilon_0)\sigma_0 - q_0 - 1\big], \tag{20}$$

and the conservation equation

$$\dot{\Lambda}_0 G_0 + 6\dot{G}_0 H_0^2\sigma_0 = 0. \tag{21}$$

In this work we adopt a method similar to that introduced by Sistero,[8] for G and Λ constants. We assume the global 'equation of state',

$$p = \frac{1}{3}\rho\phi, \tag{22}$$

where ϕ is a function of the scale factor R. This assumption makes physical sense when Eq. (22) adequately represents the cosmic content. From Eqs. (14) and (22), we obtain

$$\frac{1}{\psi}\frac{d\psi}{dR} + \frac{\phi}{R} = 0, \tag{23}$$

where

$$\psi = \rho R^3. \tag{24}$$

In Eq. (23), either ϕ or ψ may be taken to be an arbitrary function. If ϕ is a given explicit function of R, then Eq. (22) is determined, and ψ follows from Eq. (23),

$$\psi = \psi_0 \exp\Big[-\int\frac{\phi}{R}dR\Big]. \tag{25}$$

Conversely, if ψ is given, ϕ immediately follows from Eq. (23),

$$\phi = -\frac{R}{\psi}\frac{d\psi}{dR}. \tag{26}$$

The Friedman equation (13) with Eq. (24) becomes

$$3\dot{R}^2 = 8\pi G\psi R^{-1} + \Lambda R^2 - 3k. \tag{27}$$

Equations (11) and (24) with $\frac{d}{dt} = \dot{R}\Big(\frac{d}{dR}\Big)$ give

$$8\pi\frac{dG}{dR} + \psi^{-1}R^3\frac{d\Lambda}{dR} = 0. \tag{28}$$

Finally, if $G = G(R)$ is given, we integrate Eq. (28) to yield $\Lambda = \Lambda(R)$, Eq. (27) determines $R = R(t)$, and the problem is solved; $\Lambda = \Lambda(R)$ may be given instead and $G(R)$ derives from Eq. (28), giving in turn $R(t)$ from integration of Eq. (27).

As an example of matter distribution we consider a case of $\phi = 2$, a constant (numerical) in Eq. (25) thereby giving the relation

$$\psi = \rho_0\Big(\frac{R_0}{R}\Big)^3 R, \tag{29}$$

and from Eq. (24) and (29), we have

$$\rho = \rho_0 \Big(\frac{R_0}{R}\Big)^3 \frac{1}{R^2}. \tag{30}$$

Substituting the condition

$$G = G_0 \Big(\frac{R}{R_0}\Big)^n, \tag{31}$$

into Eq. (28) with ψ from Eq. (29), we have

$$\Lambda = \Lambda_0 + D_n \Big[1 - \Big(\frac{R}{R_0}\Big)^{n-5}\Big] R_0^{-2}, \tag{32}$$

where D_n is a parameter related to the integration constant of Eq. (28) and to Λ_0; its value, using Eq. (16), is given by

$$D_n = \frac{6n}{n-5} H_0^2 \sigma_0, \tag{33}$$

for $n \neq 5$.

From Eq. (19), we have

$$\Lambda_0 = 3H_0^2 \big[3\sigma_0 - q_0\big]. \tag{34}$$

Using Eqs. (29), (31) and (32), Friedman's equation (27) gives

$$\dot{R}^2 = A_n R^{n-3} + B_n R^2 - k, \tag{35}$$

where

$$A_n = \frac{-10}{n-5} H_0^2 \sigma_0 R_0^{3-n}, \tag{36}$$

$$B_n = H_0^2 \Big[\Big(3 + \frac{2n}{n-5} R_0^{-2}\Big)\sigma_0 - q_0\Big]. \tag{37}$$

Equation (20) can be written as

$$\frac{k}{R_0^2} = H_0^2 \big[5\sigma_0 - q_0 - 1\big], \tag{38}$$

and Eq. (21), i.e.,

$$\dot{\Lambda}_0 G_0 + 6 G_0 H_0^2 \sigma_0 = 0,$$

is satisfied.

It is clear that the models are completely characterized by the set of parameters (H_0, G_0, σ_0, q_0, n) with $n \neq 5$. $D_n < 0$ in Eq. (33) and $A_n > 0$ in Eq. (36) when $n < 4$ and vice versa for $n \geq 4$; if n, σ_0, and q_0 are combined with Eq. (37), then $B_n \geq (<) 0$; $\Lambda_0 \geq (<) 0$ as $\sigma_0 \geq (<) q_0$ as given by Eq. (34) and the curvature parameter k is equal to $+1$, 0, -1 according to $5\sigma_0 - q_0 - 1 \geq (<) 0$ in Eq. (38). These relations determine the integration conditions of the Friedmann equation (35) and the properties of its solutions.

In summary, Einstein's field equation is generalized with the usual conservation laws $T^{ab}_{;b} = 0$ by considering gravity with G and Λ coupling scalars. Its application to cosmology is developed. The field equations for perfect fluid cosmology are formally identical to Einstein's equations for G and Λ constants including Eq. (1). Thus the evolution of matter is similar to that of Einstein's theory as it depends on the spacetime geometry given by the scale factor $R(t)$ in Eqs. (12)–(14). The additional conservation equation (11) gives the coupling of the scalar fields with matter. We find the exact solutions of the matter distribution in cosmological models by introducing a general method of solving the cosmological field equations.

The solutions illustrate many interesting cases showing initial singularity $R(0) = 0$ with an ever-expanding or finite cosmic era returning to a future singularity $R(t) = 0$.

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