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Homoclinic Breather-Wave with Convective Effect for the (1+1)-Dimensional Boussinesq Equation *

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A new type of two-wave solution, i.e. a homoclinic breather-wave solution with convective effect, for the (1+1)-dimensional Boussinesq equation is obtained using the extended homoclinic test method. Moreover, the mechanical feature of the wave solution is investigated and the phenomenon of homoclinic convection of the two-wave is exhibited on both sides of the equilibrium. These results enrich the dynamical behavior of (1+1)-dimensional nonlinear wave fields.

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In recent years, many attempts have been made to study the nonlinear evolution equations governing wave motions in media with a damping mechanism. One of the typical examples is the Boussinesq (Bq) equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} + \sigma u_{xxxx} = 0, \quad -\infty < x < \infty, \quad t \in R, \tag{1}$$

where $\sigma > 0$ is called the "good" Bq equation and $\sigma < 0$ the "bad" Bq equation. The Bq equation is usually used to describe many real world processes such as the propagation of long waves on shallow water and the oscillations of nonlinear elastic strings. The good Bq equation describes the two-dimensional irrotational flow of an inviscid liquid in an uniform rectangular channel. There are known results due to local well-posed, global existence and blow-up of some solutions.[1−3] The bad Bq equation is used to describe two-dimensional flow of shallow-water waves having small amplitudes. There is a dense connection to the so-called Fermi-Pasta-Ulam (FPU) problem. The existence of Lax pair, Backlund transformation and some soliton-type solutions were also discussed.[3] Recently, the exact homoclinic orbit for the Bq equation was obtained and the homoclinic bifurcation of the solution was also investigated. [4−6]

In this Letter, applying a novel approach and extending the homoclinic test approach[7−13] to the (1+1)-dimensional bad Bq equation we obtain a new type of two-wave solution, i.e. a homoclinic breather-wave solution, which is a homoclinic wave homoclinic to a fixed point of Eq. (1) with one speed and meanwhile a periodic wave with another speed. Furthermore, we also investigate the mechanical feature and exhibit the homoclinic convective effect of these two-wave solutions on both sides of the equilibrium for the Bq equation.

We consider the bad Bq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0, \quad -\infty < x < \infty, \quad t \in R. \tag{2}$$

It is easy to see that Eq. (1) has an equilibrium solution u_0 which is an arbitrary constant. We suppose

$$u = u_0 + 2(\ln f)_{xx}, \tag{3}$$

where $f(x,t)$ is unknown real functions. Substituting Eq. (3) into Eq. (2), we obtain the bilinear equation[14]

$$(D_t^2 - (1+6u_0)D_x^2 - D_x^4)(f \cdot f) = 0, \tag{4}$$

where $D_x^2 f \cdot f = 2(f_{xx}f - 2f_x^2)$. In this case we choose the extended homoclinic test function

$$f = e^{-p_1(x-\alpha t)} + b_1 \cos(p(x+\beta t)) + b_2 e^{p_1(x-\alpha t)}, \tag{5}$$

where $p_1, p, \alpha, \beta, b_1$ and b_2 are real constants to be determined.

Computing $D_t^2(f \cdot f)$, $D_x^2(f \cdot f)$ and $D_x^4(f \cdot f)$ we obtain

$$\begin{aligned}
&D_t^2(f \cdot f) \\
&= 2\big[b_1(p_1^2\alpha^2 - p^2\beta^2)\cos(p(x+\beta t))e^{-p_1(x-\alpha t)} \\
&\quad + b_1 b_2(p_1^2\alpha^2 - p^2\beta^2)\cos(p(x+\beta t))e^{p_1(x-\alpha t)} \\
&\quad + 4b_2 p_1^2\alpha^2 - b_1^2 p^2\beta^2 \\
&\quad + 2b_1 p p_1 \alpha\beta \sin(p(x+\beta t))e^{-p_1(x-\alpha t)} \\
&\quad - 2b_1 b_2 p p_1 \alpha\beta \sin(p(x+\beta t))e^{p_1(x-\alpha t)}\big], \qquad (6) \\
&D_x^2(f \cdot f) \\
&= 2\big[b_1(p_1^2 - p^2)\cos(p(x+\beta t))e^{-p_1(x-\alpha t)} \\
&\quad + b_1 b_2(p_1^2 - p^2)\cos(p(x+\beta t))e^{p_1(x-\alpha t)}
\end{aligned}$$

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$$
\begin{aligned}
&+4b_2p_1^2-b_1^2p^2\\
&-2b_1pp_1\sin(p(x+\beta t))e^{-p_1(x-\alpha t)}\\
&+2b_1b_2pp_1\sin(p(x+\beta t))e^{p_1(x-\alpha t)}\big],
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&D_x^4(f\cdot f)\\
&=2\big[b_1(p_1^4+p^4-6p_1^2p^2)\cos(p(x+\beta t))e^{-p_1(x-\alpha t)}\\
&+b_1b_2(p_1^4+p^4-6p_1^2p^2)\cos(p(x+\beta t))e^{p_1(x-\alpha t)}\\
&+4b_1^2p^4+16b_2p_1^4\\
&-4b_1(p_1^3p-p^3p_1)\sin(p(x+\beta t))e^{-p_1(x-\alpha t)}\\
&+4b_1b_2(p_1^3p-p^3p_1)\sin(p(x+\beta t))e^{p_1(x-\alpha t)}\big].
\end{aligned}
\tag{8}
$$

Substituting Eqs. (6)–(8) into Eq. (4) yields an algebraic equation of $e^{p_1(x-\alpha t)}$. Equating all coefficients of different powers of $e^{jp(x-\alpha t)}$ $(j=-1,0,1)$ to zero, we obtain

$$
\begin{aligned}
&b_1(p_1^2\alpha^2-p^2\beta^2)-b_1(1+6u_0)(p_1^2-p^2)\\
&-b_1(p_1^4+p^4-6p_1^2p^2)=0,\\
&b_1b_2(p_1^2\alpha^2-p^2\beta^2)-b_1b_2(1+6u_0)(p_1^2-p^2)\\
&-b_1b_2(p_1^4+p^4-6p_1^2p^2)=0,\\
&4b_2p_1^2\alpha^2-b_1^2p^2\beta^2-(1+6u_0)(4b_2p_1^2-b_1^2p^2)\\
&-4b_1^2p^4-16b_2p_1^4=0,\\
&b_1pp_1\alpha\beta+b_1pp_1(1+6u_0)-b_1(p_1^3p-p^3p_1)=0,\\
&b_1b_2pp_1\alpha\beta+b_1b_2pp_1(1+6u_0)\\
&-b_1b_2(p_1^3p-p^3p_1)=0.
\end{aligned}
\tag{9}
$$

Solving Eqs. (9) and taking $p_1=p$ yield

$$
\begin{aligned}
p^2&=\frac{1}{4}(\beta^2-\alpha^2),\\
b_1&=\pm 2\frac{\sqrt{(2\alpha^2-(1+6u_0)-\beta^2)b_2}}{\sqrt{2\beta^2-(1+6u_0)-\alpha^2}},\\
\alpha\beta&=-(1+6u_0).
\end{aligned}
\tag{10}
$$

Choosing $u_0\neq -1/6$ and $b_2>0$, from Eq. (10) we obtain

$$
\begin{aligned}
&|\beta|>|\alpha|,\quad \alpha=-(1+6u_0)/\beta,\\
&\alpha^2>1+6u_0,\ \text{for } u_0<-1/6,\ \text{or } u_0>-1/6,\\
&\text{or } \alpha^2<1+6u_0,\ \text{for } u_0>-1/6.
\end{aligned}
\tag{11}
$$

Substituting Eqs. (10) and (11) into Eq. (3) and taking positive and negative signs in Eq. (10) yields the solutions of the bad Bq equation as follows:

$$
\begin{aligned}
u_1(x,t)=u_0+p^2\Big[&h_0+2h_1\sin(p(x+\beta t))\\
&\cdot\sinh\Big(p\Big(x+\frac{1+6u_0}{\beta}t\Big)+\gamma\Big)\Big]\\
&\cdot\Big[\cosh\Big(p\Big(x+\frac{1+6u_0}{\beta}t\Big)+\gamma\Big)\\
&+h_1\cos(p(x+\beta t))\Big]^{-2},
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
u_2(x,t)=u_0+p^2\Big[&h_0-2h_1\sin(p(x+\beta t))\\
&\cdot\sinh\Big(p\Big(x+\frac{1+6u_0}{\beta}t\Big)+\gamma\Big)\Big]\\
&\cdot\Big[\cosh\Big(p\Big(x+\frac{1+6u_0}{\beta}t\Big)+\gamma\Big)\\
&-h_1\cos(p(x+\beta t))\Big]^{-2},
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
h_0&=\frac{3(\beta^2-\alpha^2)}{2\beta^2-(1+6u_0)-\alpha^2},\\
h_1&=\frac{\sqrt{2\alpha^2-(1+6u_0)-\beta^2}}{\sqrt{2\beta^2-(1+6u_0)-\alpha^2}}<1,\\
\gamma&=\ln\sqrt{b_2},\quad p=\pm\sqrt{\beta^2-\alpha^2}/2.
\end{aligned}
$$

Now we consider the case of $u_0=-1/6$, letting $p_1^2=kp^2$, $k\neq 1$, from Eq. (9) we can reach

$$
\begin{aligned}
p^2&=\frac{\beta^2-k\alpha^2}{6k-k^2-1},\\
b_1^2&=4b_2\frac{k[(6k-k^2-1)\alpha^2-4k(\beta^2-k\alpha^2)]}{(6k-k^2-1)\beta^2+4(\beta^2-k\alpha^2)},\\
\alpha\beta&=(k-1)p^2,\quad \beta^2>k\alpha^2,\\
&3-2\sqrt{2}<k<3+2\sqrt{2},
\end{aligned}
\tag{14}
$$

and we can obtain the solution of the Bq equation as follows:

$$
\begin{aligned}
u_3(x,t)=-\frac{1}{6}+p^2\Big[&h_2+h_3(k-1)\cosh\Big(p_1\Big(x\\
&-\frac{(k-1)p^2}{\beta}t\Big)+\gamma\Big)\cos\big(p(x+\beta t)\big)^2\\
&+2h_3\sqrt{k}\sinh\Big(p_1\Big(x-\frac{(k-1)p^2}{\beta}t\Big)+\gamma\Big)\\
&\cdot\sin(p(x+\beta t))\Big]\Big[\cosh\Big(p_1\Big(x-\frac{(k-1)p^2}{\beta}t\Big)\\
&+\gamma\Big)+h_3\cos\big(p(x+\beta t)\big)\Big]^{-2},
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
h_2&=\frac{k[(10k-k^2+3)\beta^2-(10k-1+3k^2)\alpha^2]}{(6k-k^2-1)\beta^2+4(\beta^2-k\alpha^2)},\\
h_3&=\frac{\sqrt{k[(6k-k^2-1)\alpha^2-4k(\beta^2-k\alpha^2)]}}{\sqrt{(6k-k^2-1)\beta^2+4(\beta^2-k\alpha^2)}}<1,\\
\gamma&=\ln\sqrt{b_2}.
\end{aligned}
$$

The solution $u_1(x,t)$ $(u_2(x,t))$ shows a new type of two-wave, i.e. homoclinic breather-wave solution, which is a homoclinic wave homoclinic to a fixed point u_0 of Eq. (2) when $t\to\pm\infty$,[4,5] and meanwhile is a periodic wave whose amplitude periodically oscillates with the evolution of time. It shows elastic interaction between a left-propagation (backward-direction) periodic wave with speed β and a homoclinic wave

of different directions with speed $(1+6u_0)/\beta$. The structure of the solution in an arbitrary small domain of $u_0 = -1/6$ is different, the homoclinic wave moves with the backward-direction (left-propagation) when u_0 lies to the right side of $-1/6$, $(u_0 > -1/6)$, while this waves moves with the forward-direction (right-propagation) when u_0 lies to the left side of $-1/6$, $(u_0 < -1/6)$. In both the above cases, the width of the two waves is the same and the propagation speed is different (Fig. 1). Moreover, when $u_0 = -1/6$, the mechanical feature of the homoclinic breather-wave is different from the case of $u_0 \neq -1/6$ in which the homoclinic wave with width p_1^{-1} and the periodic wave with width p^{-1} move along the same direction when $3-2\sqrt{2} < k < 1$ or along the opposite direction when $1 < k < 3+2\sqrt{2}$. All these different features in an arbitrary small domain of $u_0 = -1/6$ show a homoclinic convective effect for the two-wave at the equilibrium $u_0 = -1/6$. This is a strange and interesting physical phenomenon to the evolution of the two-dimensional flow of shallow-water waves having small amplitudes. We will study this interesting phenomenon in further work.

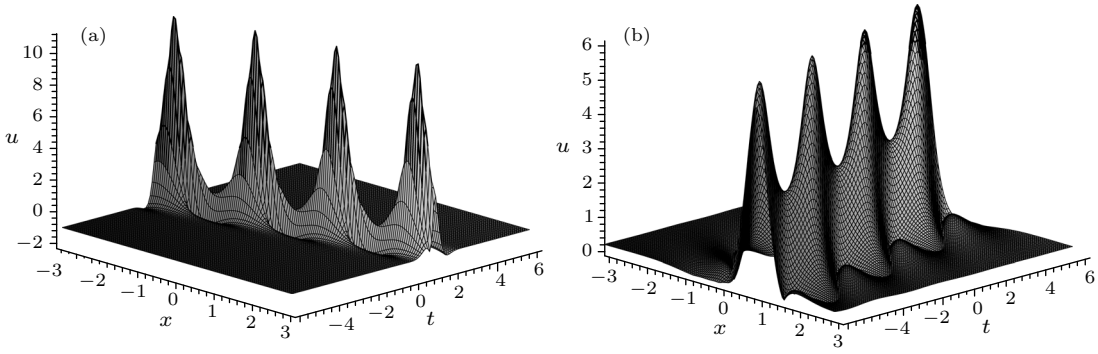

Fig. 1. Behavior of $u_1(x,t)$: (1) $b_2 = 1$, $u_0 = -1$, $\alpha = 1$, $\beta = 2$; (2) $b_2 = 1$, $u_0 = -1/7$, $\alpha = \sqrt{3/5}$, $\beta = 1$.

In summary, applying an extended homoclinic test method to the good Bq equation and by the same process of dealing with Eq. (2), we obtain an analogical result. However, the mechanical feature of the homoclinic breather wave is different from the bad Bq equation.

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