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Reactive Coupling Effects on Amplitude Death of Coupled Limit-Cycle Systems *

WANG Jin-Hua(王金华), LI Xiao-Wen(李晓文)**
Department of Physics, Beijing Normal University, Beijing 100875

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Amplitude death in coupled limit-cycle systems induced by the reactive coupling is studied. The presence of reactive coupling parameter changes the amplitude death process of the system, and increases the critical coupling strength for the emergence of amplitude death. When the systems are in the state of complete or partial amplitude death, increasing the reactive coupling will increase the number of partial synchronization groups, implying the increase of disorder of the system. Increasing the reactive coupling makes the elimination of the amplitude death of the systems harder.

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Coupled limit-cycle systems provide simple but useful mathematical models for researching the collective behaviour and spatiotemporal patterns of a wide variety of systems in physical, chemical and biological fields.[1−7] Amplitude death is a coupling-induced stabilization of the system:[8] the oscillators pull each other off their limit cycles and collapse into a state of zero amplitude. Yamaguchi and Shimizu found that two ingredients are needed for amplitude death to be stable: sufficiently strong coupling between the oscillators and a sufficiently wide distribution of natural frequencies.

In this Letter, we study the dynamics of a coupled limit-cycle system with nonlinear frequency pulling and reactive coupling, which can be described by the following equations of motion:[8−10]

$$\begin{aligned}\dot{Z}_j =& i(\omega_j - \alpha|Z_j|^2)Z_j + (1-|Z_j|^2)Z_j \\ &+ \frac{K+i\beta}{3}(Z_{j+1}+Z_{j-1}-2Z_j), \\ & j = 1, 2, \ldots N, \end{aligned} \tag{1}$$

where Z_j is a complex variable which represents the state of the jth oscillator: $Z_j = r_j e^{i\theta_j}$; $\{\omega_j\}$ are the natural frequencies of the oscillators. The frequencies are assumed to obey the uniform distribution, $\omega_j \in \left[-\frac{\Delta\omega}{2}, \frac{\Delta\omega}{2}\right]$; α is the nonlinear frequency pulling; $K \geq 0$ is the coupling strength, and β is the reactive coupling. Here we study the case of periodic boundary condition and discuss the case of zero frequency pulling, i.e., $\alpha = 0$.

In order to investigate the collective behaviour of the system, we give two quantities,[11] i.e., the normalized mean 'incoherent energy'

$$E = \left\langle \sum_{j=1}^{N} |Z_j|^2 \right\rangle \Big/ N, \tag{2}$$

and the percentage of partial amplitude death

$$P_a = \langle N_a \rangle / N, \tag{3}$$

where a pair of brackets represents the averaging over time. N_a is the number of active oscillators with nonzero amplitude. $P_a = 1$ corresponds to the case of no amplitude death. $0 < P_a < 1$ refers to the presence of partial amplitude death. When $E = 0$ and $P_a = 0$, the system comes to complete amplitude death.

We consider the case $N = 1000$, $\beta = 0.0$. Let us first investigate the dependence of E and P_a on coupling strength for different $\Delta\omega$. In Figs. 1(a) and 1(b), for $\Delta\omega = 10$, E and P_a never reach zero with the increase of the coupling strength, indicating that for small $\Delta\omega$, global amplitude death is impossible. For $\Delta\omega = 30$, E and P_a reach zero when K is large enough, and complete amplitude death exists in a range of the coupling strength for a large $\Delta\omega$.

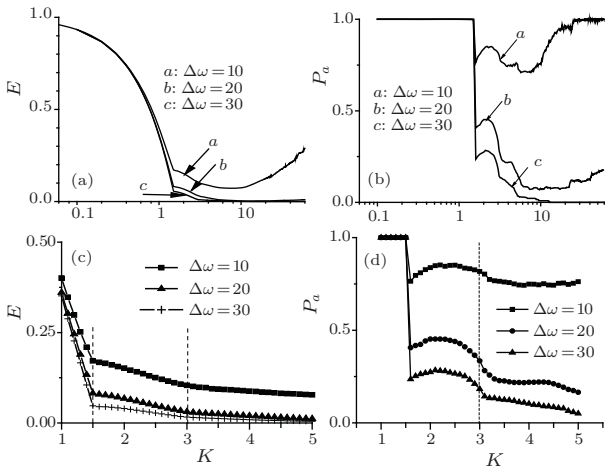


Fig. 1. $N = 1000$, for different $\Delta\omega$, (a) dependence of E on the coupling strength K, (b) dependence of P_a on K, (c) and (d) the magnification of (a) and (b) for $K \in [1.0, 5.0]$.

From Figs. 1(c) and 1(d) we know that there are three stages during the transition to global amplitude death. In the first stage where $K < 1.5$, for different $\Delta\omega$, $P_a = 1$ is held. This means that all oscillators are active. E decreases with the increasing K, which means that even when no oscillator becomes dead, the

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**To whom correspondence should be addressed. Email: xwli@bnu.edu.cn
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amplitude of the oscillators decreases. While $K \approx 1.5$, P_a drops sharply to a small value from 1, indicating that some oscillators become dead. The second stage ranges from $K \approx 1.5$ to $K \approx 3.0$. For different $\Delta\omega$, with the increase of K, E and P_a increase firstly then decrease. The larger the $\Delta\omega$, the less the P_a, implying that the larger the $\Delta\omega$, the larger the number of dead oscillators. The third stage is $K \geq 3.0$. From Figs. 1(a) and 1(b) we know that, when $\Delta\omega$ is large enough, the complete amplitude death can come true. When $\Delta\omega$ is small, the system can not become complete amplitude death.

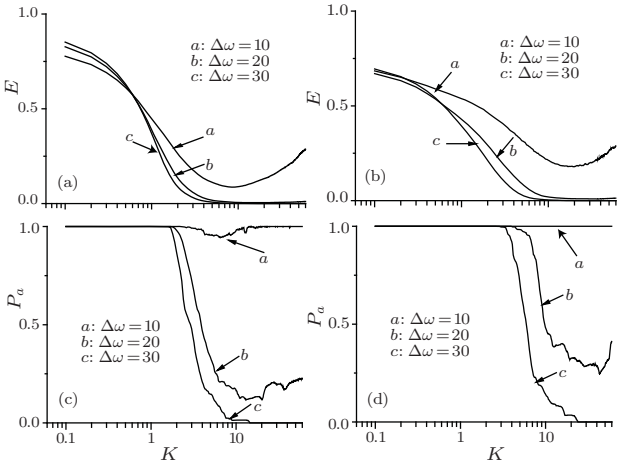


Fig. 2. $N = 1000$, dependences of E on the coupling strength K: (a) $\beta = 3.0$, (b) $\beta = 10.0$. Dependences of P_a on K: (c) $\beta = 3.0$, (d) $\beta = 10.0$.

Then we investigate the effects of β on the amplitude death of the systems. The results are presented in Figs. 2. Figures 2(a) and 2(b) show that β can change the stages of amplitude death. With the increase of β, the effects increase too. Figures 2(c) and 2(d) give the dependence of P_a on K for $\beta = 3.0$ and $\beta = 10.0$. With the increase of K, P_a decreases smoothly from 1, indicating that the number of dead oscillators increases gradually with the increase of coupling strength. This is different from the sharp change of P_a in Fig. 1(b), where some oscillators become dead collectively at $K \approx 1.5$. In comparison of the changes of P_a for $\Delta\omega = 10.0$ in Figs. 2(c) and 2(d), in a range of the coupling strength, the system can become partial amplitude death when $\beta = 3.0$ while it can not become partial amplitude death when $\beta = 10.0$. The larger the β, the harder the death of oscillators. In Figs. 2(c) and 2(d), for $\Delta\omega = 30.0$, the critical coupling strength for the system to become complete death becomes larger when β increases.

We define two critical coupling strengths, K_{c1} and K_{c2}. K_{c1} means the appearance of partial amplitude death of the system. The corresponding P_a changes from 1 to a small value. K_{c2} means the appearance of global amplitude death of the system. The corresponding P_a changes from a nonzero value to zero. Figure 3 gives the dependence of K_{c1} and K_{c2} on the reactive coupling parameter when $\Delta\omega = 35$. With the import of β, both K_{c1} and K_{c2} become larger. The larger the β, the larger the critical coupling strength.

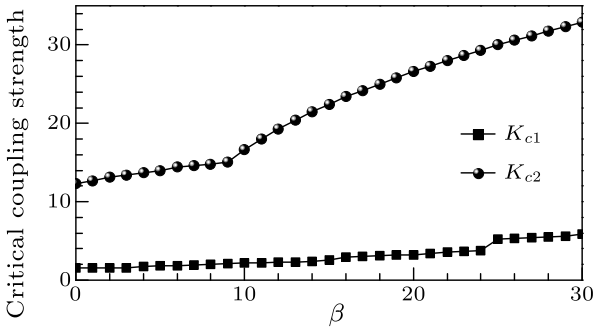


Fig. 3. $N = 1000$, $\Delta\omega = 35$, dependence of the critical coupling strength K_{c1} and K_{c2} on β.

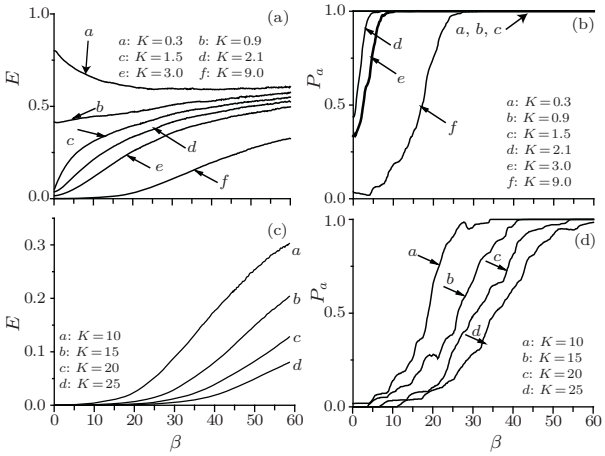


Fig. 4. $N = 1000$, $\Delta\omega = 30$, for different K: (a) and (c) dependences of E on β, (b) and (d) dependences of P_a on β.

Without considering β, when the coupling strength is large enough, the systems are in complete amplitude death or partial amplitude death. We investigate the effect of β on the state of the system underside. Figures 4(a) and 4(c) show the dependence of E on the reactive coupling β when $\Delta\omega = 30$. When the coupling strength K is large enough, increasing β can augment E, indicating the augmentation of the amplitude of oscillators. Figures 4(b) and 4(d) show the dependence of P_a on the reactive coupling β when $\Delta\omega = 30$. When the coupling strength K is large enough and $\beta = 0$, the systems are in complete amplitude death state or partial amplitude death state. The augmentation of β eliminates the complete or partial amplitude death states. Comparing Figs. 4(b) with 4(d), we discover that the larger the coupling strength, the larger the reactive coupling parameter for P_a to change from a decimal fraction to 1. This means that the larger the coupling strength, the harder the elimination of amplitude death completely.

When $\alpha = 0$, the dynamical equation of the system is

$$\dot{Z}_j = (i\omega_j + 1 - |Z_j|^2)Z_j + \frac{1}{3}(K + i\beta)(Z_{j+1} + Z_{j-1} - 2Z_j). \quad (4)$$

In order to observe the effect of β on the system, we change the oscillation of every oscillator into polar coordinates $Z_j = r_j e^{i\theta_j}$. Then we obtain

$$\begin{aligned}\dot{r}_j &= (1-r_j^2)r_j + \frac{K}{3}\big(r_{j+1}\cos(\theta_{j+1}-\theta_j) \\ &\quad + r_{j-1}\cos(\theta_{j-1}-\theta_j) - 2r_j\big) - \frac{\beta}{3}\big(r_{j+1} \\ &\quad \cdot \sin(\theta_{j+1}-\theta_j) + r_{j-1}\sin(\theta_{j-1}-\theta_j)\big), \qquad (5) \\ \dot{\theta}_j &= \omega_j + \frac{K}{3r_j}\big(r_{j+1}\sin(\theta_{j+1}-\theta_j) \\ &\quad + r_{j-1}\sin(\theta_{j-1}-\theta_j)\big) + \frac{\beta}{3r_j}\big(r_{j+1}\cos(\theta_{j+1}-\theta_j) \\ &\quad + r_{j-1}\cos(\theta_{j-1}-\theta_j) - 2r_j\big). \qquad (6)\end{aligned}$$

From the above equations, we can see that the effects of β on the system is similar to that of the phase-shifted coupling of the system. The influence of phase-shifted coupling has been studied by Zheng in Ref. [12]. The conclusion is as follows: the phase-shifted coupling can eliminate the complete synchronous state or partial synchronous state of the system. That can explain β makes the amplitude death of the systems harder. In order to validate the effects of β on the average frequencies of the systems, we will study the average frequencies of the system underside.

We define the average frequency of the oscillators as

$$\Omega_j = \lim_{T\to\infty}\frac{1}{T}\int_0^T \dot{\theta}_j(t)dt. \qquad (7)$$

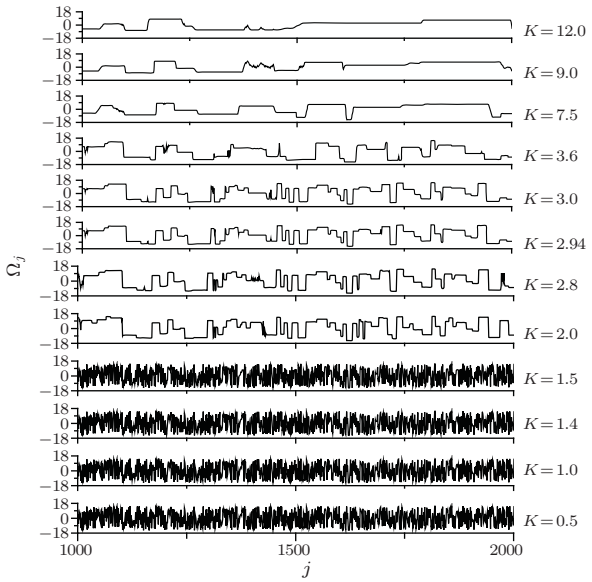


Fig. 5. $N = 10000$, $\Delta\omega = 30$,$\beta = 0.0$, dependence of the 1000th to 2000th oscillators' average frequencies on coupling strength.

In order to observe the diversifications of the average frequencies, we set $N = 10000$. Figure 5 shows the 1000th to 2000th oscillators' average frequencies. Every oscillator oscillates with its natural frequency when the coupling strength is small. With the increase of the coupling strength, some small synchronous groups come forth, indicating that the system comes into partial synchronization. Keeping on increasing the coupling strength, the number of synchronous groups is decreasing. From Fig. 1(b) we know that when the coupling strength is small, the oscillators are all active. Increasing the coupling strength, some oscillators become amplitude death. Keeping on increasing the coupling strength, the number of dead oscillators becomes larger. When the coupling strength is large enough, the system will come to complete amplitude death. Thus we can educe a conclusion that the larger the coupling strength, the more orderly the system. The order of the system accelerates the amplitude death of the system.

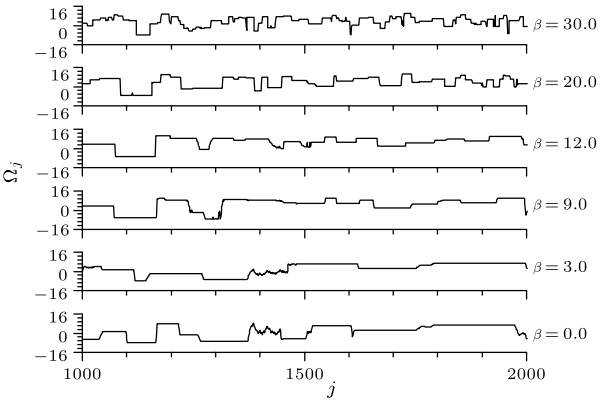


Fig. 6. $N = 10000$, $\Delta\omega = 30$, $K = 9.0$, dependence of the average frequencies on β.

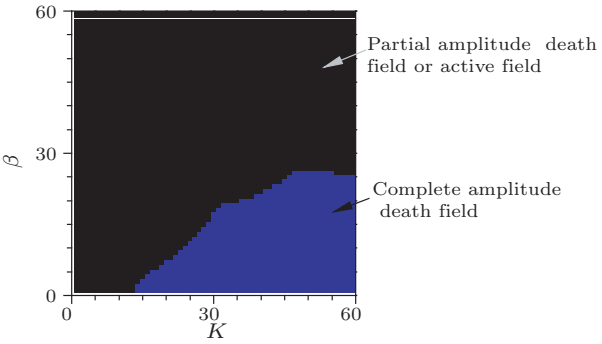


Fig. 7. Amplitude death area in the K and β plane.

Figure 6 shows the dependence of the average frequencies of the 1000th to 2000th oscillators on the reactive coupling β. With the increase of β, the number of the synchronous groups becomes larger. Thus the more disorder the system is. From Fig. 4 we know, increasing β activates the dead oscillators of the system. From the point of the average frequencies of the oscillators we know the increase of the disorder of the system eliminates the amplitude death of the system.

We can conclude that when the span of the natural frequencies is large enough, increasing the coupling strength can accelerate the amplitude death, while increasing the reactive coupling can decelerate the amplitude death or eliminate the amplitude death of the

system. Thus the two parameters of the system have opposite effects on the amplitude death of the system. Figure 7 gives the amplitude death area in K and β plane. The import of β eliminates the amplitude death of the system.

Figure 8 shows the dependence of amplitudes of the 1000th to 2000th oscillators on the coupling strength. While $K = 0.5$, all the amplitudes of the oscillators distribute around 1. When $K \approx 1.5$, some oscillators become dead collectively. Keeping on increasing the coupling strength, the number of dead oscillators becomes larger. When the coupling strength is large enough, the system will come to global death.

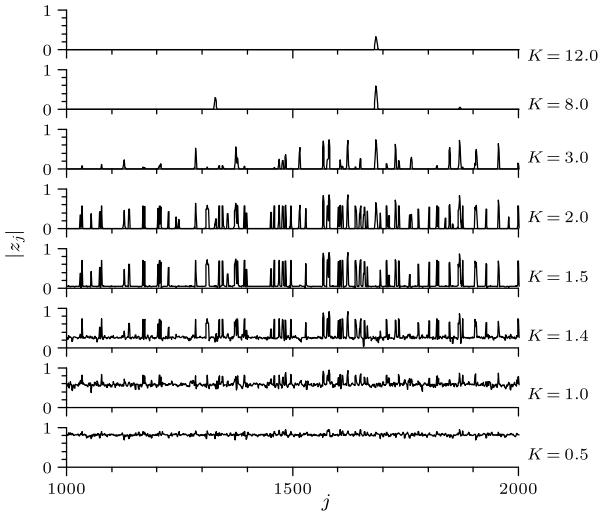


Fig. 8. $N = 10000$, $\Delta\omega = 30$, $\beta = 0$, dependence of amplitudes on the coupling strength.

According to the above investigation, reactive coupling can change the stage of amplitude death. We will study the dependence of the amplitude of the oscillators by considering the reactive coupling underside. Figure 9 gives the dependence of the amplitudes on the parameters under $\beta = 3.0, \Delta\omega = 30$. While $K = 0.5$, all the oscillators distribute around 1, but the distribution is more dispersive compared with that of $\beta = 0$. As the coupling strength increases, the distribution becomes even more dispersive. While $K \approx 2.0$, some oscillators become dead. Increasing the coupling strength can increase the number of dead oscillators. From the above investigation one can conclude, with the import of reactive coupling, the oscillators' distribution becomes dispersive and the critical coupling strength for the system to partial amplitude death is large. The stages of amplitude death are altered, too. When $\beta = 3.0$, the number of dead oscillators is increased gradually. When $\beta = 0.0$, the number of dead oscillators changes from 1 to a small value suddenly at the critical coupling strength $K \approx 1.5$.

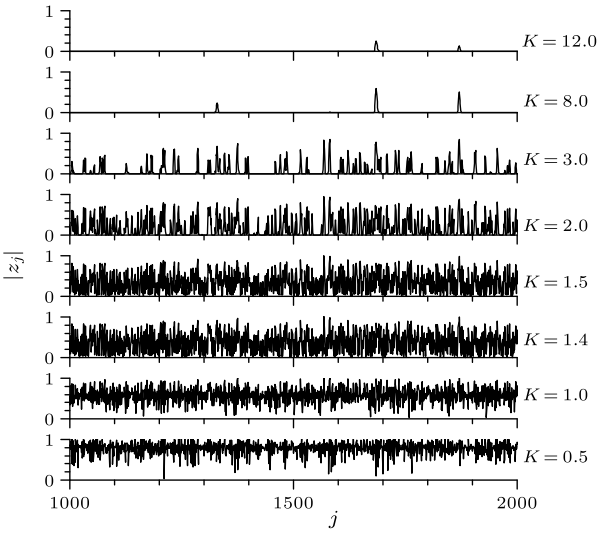


Fig. 9. $N = 10000$, $\Delta\omega = 30$, $\beta = 3.0$, dependence of amplitudes on the coupling strength.

We study the amplitude death phenomena of the coupled limit-cycle systems by considering reactive coupling. Using the normalized average incoherent energy and the percentage of active oscillators, we distinctly find out the effects of the reactive coupling on the amplitude death of the systems. The reactive coupling makes the number of dead oscillators increase gradually, the critical coupling strength for the amplitude death of the systems become greater, and even eliminates the amplitude death of the systems under some special conditions. The phase dynamics indicates that the import of the reactive coupling increases the disorder of the system and makes the amplitude death of the system harder. From the point of the amplitude dynamics, we expressly know the changes of the stages of the amplitude death of the system and the effects of the reactive coupling.

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