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Calculation of Covariance Matrix for Multi-mode Gaussian States in Decoherence Processes *

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We investigate the dynamics of n single-mode continuous variable systems in a generic Gaussian state under the influence of the independent and correlated noises making use of the characteristic function method. In two models the bath is assumed to be a squeezed thermal one. We derive an explicit input-output expression between the initial and final covariance matrices. As an example, we study the evolution of entanglement of three-mode Gaussian state embedded in two noisy models.

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Quantum entanglement has been an important physical resource in the emerging field of quantum information science.[1,2] However, in a realistic implementation, quantum entanglement must be distributed and shared among many communication parties in advance along physical channels, such as optical fibres and the atmosphere. Due to inevitable interactions with the environments concerned, the distributed state will decohere and lose its initial entanglement, thus leading to deterioration in the performance of quantum information tasks and even to the failure of quantum information protocols. In addition, this quantum decoherence process is important in the understanding of the quantum-classical transition.[3−5] Therefore, studying the dynamics of an open quantum system is of great value and extensive investigations have been performed.[6−8]

A Gaussian continuous variable (CV) entangled states, as an alternative to the entangled qubit states, have recently stimulated to a lot of interest due to the relative simplicity of and the high efficiency achieved in the production,[9] manipulation,[10] and detection of optical Gaussian CV states. These states have already been used in areas such as quantum teleportation,[11−13] quantum entanglement swapping[14] and quantum key distribution.[15] It is therefore important to know how the interaction with an environment influences the dynamic behaviour of entanglement of these Gaussian states. In recent years, decoherence and entanglement degradation of a Gaussian state in continuous variable systems have spurred several theoretical works. In particular, the effect of a variety of decoherence mechanisms on the bipartite two-mode Gaussian states of continuous variable systems has attracted much attention.[16−20] This is because the quantification of entanglement of bipartite Gaussian states (pure and mixed) has been well understood.[21] However, up to now, a manageable, necessary and sufficient entanglement criterion for a general case of a Gaussian multipartite state is still lacking, much research has been carried on studying the dynamics of the multipartite multimode Gaussian state in different decoherece processes over the last few years.[22,23] On the other hand, it has been shown that the purity and entanglement of a Gaussian state are uniquely determined by its covariance matrix.[24,25] Thus, in a sense, an evolved Gaussian state can be described by the influence of a noisy environment over the covariance matrix of this state. Nevertheless, we find that the decoherence of a generic multimode Gaussian state has not yet been investigated under a dissipative environment.

Let us now consider the case of the n-mode bosonic fields embedded in independent and correlated noises. These kinds of decoherence models are common in several physical systems and have been extensively investigated.[22,26−28] For the first model, we consider a system of n single-mode harmonic oscillators described by the Hamiltonians $\hat{H}_s^{\otimes j}$. Each of them is independently coupled to its own squeezed reservoirs described by the Hamiltonians $\hat{H}_R^{\otimes i}$. Each bath is assumed to consist of a set of independent harmonic oscillators with the creation Γ_i^+ and annihilation Γ_i. Thus the Hamiltonian in the interaction picture describing quantum system plus reservoir is given by (setting $\hbar = 1$)

$$H_I(t) = \sum_{j=1}^{n} \Big[a_j^+ \Gamma_i e^{i\omega_0 t} + \text{H.c.} \Big], \tag{1}$$

where $\Gamma_i = \sum_i g_i b_i e^{-i\omega_i t}$ and $[b_j, b_k^+] = \delta_{jk}$.

The reservoir of interest is assumed to be a general

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squeezed thermal bath with the correlations[29]

$$\langle b_j(\omega)b_j(\omega')\rangle = 2\pi M_j(\omega)\delta(2\omega_0-\omega-\omega'), \quad (2a)$$

$$\langle {b_j}^+(\omega)b_j(\omega')\rangle = 2\pi N_j(\omega)\delta(\omega-\omega'), \quad (2b)$$

$$\langle b_j(\omega)b_j^+(\omega')\rangle = 2\pi[N_j(\omega)+1]\delta(\omega-\omega'), \quad (2c)$$

where $N_i(\omega)$ is the mean photon number of the ith squeezed reservoir, and $M_i(\omega)$ is a parameter related to the phase correlations of the ith squeezed reservoir. They are related to the squeezing parameter r_i and its phase θ_i relative to the external field by $N_i(\omega)=\sinh^2 r_i$ and $M_i(\omega)=\frac{1}{2}e^{i\theta_i}\sinh(2r_i)$. In general the positivity of the density operator requires, $|M_i(\omega)| \le \sqrt{N_i(\omega)(N_i(\omega)+1)}$. For convenience, we suppress the functional dependence of $N_i(\omega)$ and $M_i(\omega)$.

Under the Born–Markov approximation, the master equation for n single-mode fields can be written as

$$\begin{aligned}\frac{\partial}{\partial t}\rho =& \sum_{i=1}^{n}\Big[\frac{\gamma_i}{2}N_i\Big(2a_i^+\rho a_i - a_i a_i^+\rho - \rho a_i a_i^+\Big)\\ &+\frac{\gamma_i}{2}(N_i+1)\Big(2a_i\rho {a_i}^+ - a_i^+a_i\rho - \rho a_i^+a_i\Big)\\ &+\frac{\gamma_i}{2}{M_i}^*\Big(2a_i\rho a_i - \rho a_i a_i - a_i a_i\rho\Big)\\ &+\frac{\gamma_i}{2}M_i\Big(2a_i^+\rho a_i^+ - a_i^+a_i^+\rho - \rho a_i^+a_i^+\Big)\Big], \quad (3)\end{aligned}$$

where γ_i is the damping rate of the ith mode.

The time evolution of the density matrix can be obtained by solving Eq. (3). However, we have chosen to work with the characteristic function (CF) because it is more convenient for the problem under consideration. The characteristic function of n-mode fields is defined in Wigner representation as [30]

$$\chi_n(\beta) = \mathrm{Tr}\Big[\rho\prod_{i=1}^{n}\exp\Big(\beta_i a_i^+ - \beta_i^* a_i\Big)\Big], \quad (4)$$

where $\chi_n(\beta)=\chi(\beta_1,\beta_1^*,\cdots,\beta_n,\beta_n^*)$.

Using standard operator correspondence yields the following equation for the characteristic function

$$\frac{\partial}{\partial t}\chi_n(\eta,t) = \sum_{i=1}^{n}(A_i+B_i)\chi_n(\beta,t), \quad (5)$$

where we have chosen $A_i = -\frac{\gamma_i}{2}\big[(1+2N_i)|\beta_i|^2 + M_i\beta_i^{*2} + M_i^*\beta_i^2\big]$ and $B_i = -\frac{\gamma_i}{2}\Big(\beta_i^*\frac{\partial}{\partial\beta_i^*} + \beta_i\frac{\partial}{\partial\beta_i}\Big)$. It is not difficult to verify that they satisfy the commutation relation: $[A_i,B_i]=\gamma_i A_i$. Furthermore, we can write the formal solution of Eq. (5) as

$$\chi_n(\beta,t) = \prod_{i=1}^{n}\exp\Big\{\frac{A_i}{\gamma_i}\Big(1-e^{-\gamma_i t}\Big)\Big\}e^{B_i t}\chi_n(\beta,0), \quad (6)$$

where $\chi_n(\beta,0)$ is the initial characteristic function of n modes.

Assuming that the system are initially prepared in an n-mode Gaussian state with the covariance matrix (CM) of the form

$$\boldsymbol{V}_n(0) = \begin{pmatrix} n_1 & 2m_1 & m_{s12} & m_{c12} & \cdots & m_{s1n} & m_{c1n}\\ 2m_1^* & n_1 & m_{c12}^* & m_{s12}^* & \cdots & m_{c1n}^* & m_{s1n}^*\\ m_{s12}^* & m_{c12} & n_2 & 2m_2 & \cdots & m_{s2n} & m_{c2n}\\ m_{c12}^* & m_{s12} & 2m_2^* & n_2 & \cdots & m_{c2n}^* & m_{s2n}^*\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ m_{s1n}^* & m_{c1n} & m_{s2n}^* & m_{c2n} & \cdots & n_n & 2m_n\\ m_{c1n}^* & m_{s1n} & m_{c2n}^* & m_{s2n} & \cdots & 2m_n^* & n_n \end{pmatrix}. \quad (7)$$

Substituting Eq. (7) into Eq. (6), we can obtain the time evolution of the characteristic function, $\chi_n(\beta,t) = \exp\big\{-\frac{1}{4}\Lambda^T\boldsymbol{V}_n(t)\Lambda\big\}$, where $\boldsymbol{V}_n(t)$ is a $2n\times 2n$ CM given by

$$\boldsymbol{V}_n(t) = T[\boldsymbol{V}_n(0)-R]T + R, \quad (8)$$

the matrices are $T=\oplus_{i=1}^{n}\mathrm{diag}\big(e^{-\gamma_i t/2}, e^{-\gamma_i t/2}\big)$ and $R=\oplus_{i=1}^{n}R_i$, where R_i is given by

$$R_i = \begin{pmatrix} 2N_i+1 & 2M_i\\ 2M_i^* & 2N_i+1\end{pmatrix}. \quad (9)$$

Thus, the time-dependent CM of an n-mode Gaussian state is directly calculated by means of formula (8) once the initial $\boldsymbol{V}_n(0)$ and the parameters of the external environment are known.

For the second noisy model, let us recall a simple case of two single-mode fields coupled to a common source: squeezed bath. The supposition is that the fields are spatially so far apart that there is no interaction between them. The master equation of motion for the density operator of the two modes is given, in the interaction picture, by[28]

$$\begin{aligned}\frac{\partial}{\partial t}\rho =& \frac{\gamma}{2}N\sum_{i=1,2}\big(2a_i^+\rho a_i - a_i a_i^+\rho - \rho a_i a_i^+\big)\\ &+\frac{\gamma}{2}(N+1)\sum_{i=1,2}\big(2a_i\rho {a_i}^+ - a_i^+a_i\rho - \rho a_i^+a_i\big)\\ &+\gamma M\big(a_1\rho a_2 + a_2\rho a_1 - a_1a_2\rho - \rho a_1a_2\big)\\ &+\gamma M^*\big(2a_1^+\rho a_2^+ + a_2^+\rho a_1^+ - a_1^+a_2^+\rho - \rho a_1^+a_2^+\big), \quad (10)\end{aligned}$$

where γ represents the damped constant of the two modes of the fields in the correlated environment, N and M have the same physical meanings as before.

Application of the above case to the multimode-system case is straightforward. In this case, the master equation can be easily obtained as

$$\begin{aligned}\frac{\partial}{\partial t}\rho =& \frac{\gamma}{2}N\sum_{i=1}^{n}\big(2a_i^+\rho a_i - a_i a_i^+\rho - \rho a_i a_i^+\big)\\ &+\frac{\gamma}{2}(N+1)\sum_{i=1}^{n}\big(2a_i\rho {a_i}^+ - a_i^+a_i\rho - \rho a_i^+a_i\big)\\ &+\gamma M^*\sum_{i\neq j=1}^{n}\big(a_i^+\rho a_j^+ - a_i^+a_j^+\rho\big)\end{aligned}$$

$$+\gamma M\sum_{i\neq j=1}^{n}\left(a_i\rho a_j - a_i a_j\rho\right). \tag{11}$$

Using the same procedure, we can obtain the evolution of the characteristic function of the n-mode system as

$$\begin{aligned}\frac{\partial}{\partial t}\chi_n(\beta,t)&\\ =&-\frac{\gamma}{2}\Big\{\sum_{i=1}^{n}\Big[\beta_i\frac{\partial}{\partial\beta_i}+\beta_i^*\frac{\partial}{\partial\beta_i^*}+(2N+1)|\beta_i|^2\Big]\\ &+2\sum_{i\neq j=1}^{n}\left(M^*\beta_i^*\beta_j^*+M\beta_i\beta_j\right)\Big\}\chi_n(\beta,t).\end{aligned} \tag{12}$$

If the n-mode field is initially in the state with the CM (7), then it is straightforward to calculate the CF at any time t as $\chi_n(\beta,t)=\exp\{-\frac{1}{4}\Lambda^T\boldsymbol{\sigma}_n(t)\Lambda\}$ with the $2n\times 2n$ covariance matrix $\boldsymbol{\sigma}_n(t)$ given by

$$\boldsymbol{\sigma}_n(t)=T[\boldsymbol{\sigma}_n(0)-R]T+R, \tag{13}$$

where $T=e^{-\gamma t/2}\oplus_{i=1}^{n}\mathrm{diag}(1,1)$ and

$$\boldsymbol{R}=\begin{pmatrix}\boldsymbol{\alpha}&\boldsymbol{\beta}&\cdots&\boldsymbol{\beta}\\ \boldsymbol{\beta}&\boldsymbol{\alpha}&\cdots&\boldsymbol{\beta}\\ \vdots&\vdots&\ddots&\vdots\\ \boldsymbol{\beta}&\boldsymbol{\beta}&\cdots&\boldsymbol{\alpha}\end{pmatrix},\ \boldsymbol{\beta}=\begin{pmatrix}0&M\\ M^*&0\end{pmatrix}, \tag{14}$$

where $\boldsymbol{\alpha}=(2N+1)\mathrm{I}$, which is again one of the main results in this study. It allows us to easily work out the covariance matrix of a multi-mode Gaussian state affected by the influence of a correlated noisy at any time.

Finally, we investigate the entanglement properties of n-mode pure Gaussian states under the influence of the above-mentioned environment. It is important to notice that when the environment under consideration is viewed as a linear and Gaussian channel, the input Gaussian state will thus remain Gaussian in later transmission. For simplicity, the situation of $\gamma_i=\gamma_j=\gamma$ will be considered in the following. On the other hand, to date, the problem of evaluating Gaussian measures of entanglement for a multipartite Gaussian state has not yet been solved. Thus, our study is restricted to a case in which only three modes are involved. The CM of such a three-mode Gaussian state can be written as

$$\boldsymbol{V}_3(0)=\begin{pmatrix}\boldsymbol{\varepsilon}&\boldsymbol{m}&\boldsymbol{m}\\ \boldsymbol{m}&\boldsymbol{\varepsilon}&\boldsymbol{m}\\ \boldsymbol{m}&\boldsymbol{m}&\boldsymbol{\varepsilon}\end{pmatrix}, \tag{15}$$

where $\boldsymbol{\varepsilon}=\cosh(2r)\mathrm{I}$ and $\boldsymbol{m}=\sinh(2r)\sigma^x$ with r being the degree of squeezing. In fact, Eq. (15) is said to be a fully symmetric n-mode Gaussian state.[25]

To utilize the known results associated to a Gaussian state, we need to perform a transformation $\beta_i=(x_i+p_i)/\sqrt{2}$. On substituting $\boldsymbol{V}_3(0)$ into Eqs. (8) and (13), we can obtain the same expression of the CF $\exp\{-\frac{1}{4}x\boldsymbol{V}_3(t)x^T\}$ with the CM given by

$$\boldsymbol{V}_3(t)=\begin{pmatrix}e_+&0&a&0&a&0\\ 0&e_-&0&-a&0&-a\\ a&0&e_+&0&a&0\\ 0&-a&0&e_-&0&-a\\ a&0&a&0&e_+&0\\ 0&-a&0&-a&0&e_-\end{pmatrix}, \tag{16}$$

where $e_\pm=e^{-\gamma t}\cosh(2r)+(1-e^{-\gamma t})[1+2(N\pm M)]$ and $a=e^{-\gamma t}\sinh(2r)$ for independent noise, while for correlated noise, $e_+=e_-=e^{-\gamma t}\cosh(2r)+(1-e^{-\gamma t})[1+2N]$ and $a=e^{-\gamma t}\sinh(2r)+M(1-e^{-\gamma t})$.

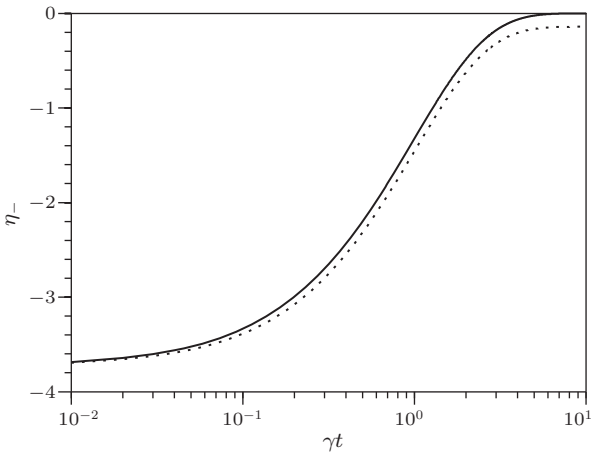

Fig. 1. Time evolution of the negativity η_- of the tripartite continuous variables affected by a independent noisy (solid line) and correlated noisy (dashed line) for the case of an initial three-mode squeezed vacuum state with a squeezing parameter $r=1$.The other parameters used are $N=4, M=2\sqrt{5}$, and $\theta=0$.

We can use an inseparability criterion based on the partial transpose (NPT) of the composite density operator,[31] which is a sufficient and necessary condition for the inseparability of $1\times N$ bipartite Gaussian system, in order to check whether a state is entangled or not. This criterion was later shown by Giedke[32] to be a necessary and sufficient condition for separability associated with three modes. Up to a three-mode Gaussian state with the Wigner characteristic function in Eq. (16), the NPT criterion states that the state is bipartite inseparable between the jth mode and the group of the rest two modes if and only if $\Gamma_j\boldsymbol{M}_3(t)\Gamma_j+i\Lambda<0$, where $\Lambda=\oplus_{j=1}^{3}i\sigma_j^y$, $\Gamma_1=\mathrm{diag}(1,-1,1,1,1,1)$, $\Gamma_2=\mathrm{diag}(1,1,1,-1,1,1)$, and $\Gamma_3=\mathrm{diag}(1,1,1,1,1,-1)$. In addition, thanks to the fact that the correlation matrix $\boldsymbol{M}_3(t)$ in Eq. (16) is permutational symmetry among the three modes. Thus, a three-mode Gaussian state is fully inseparable (genuine three-mode entanglement) if and only if the condition $\Gamma_j\boldsymbol{M}_3(t)\Gamma_j+i\Lambda<0$ is simultaneously satisfied for $(j=1,2,3)$, resulting from that the negative eigenvalues of the matrix $\Gamma_j\boldsymbol{M}_3(t)\Gamma_j+i\Lambda$ are the same for $(j=1,2,3)$ and denoted by η_-. The nega-

tivity η_- characterizes the entanglement properties of the three-mode Gaussian state, and hence we utilize the negativity η_- to quantify the genuine three-mode entanglement of the present models.

Figure 1 displays the time evolution of the negativity η_-. It can be seen that in two noisy models, the negativity η_- rapidly increases with increasing transmission time, implying that the initial entanglement of three modes will lose. However, for correlated noise, the negativity η_- does not equal to zero but saturates to a constant only determined by the bath over the long-time limit. It shows that the fields evolve into a robust three-mode Gaussian entangled mixed state instead of three-mode classical mixed one.

In summary, the equations of characteristic function of n single-mode fields are investigated, in order to elucidate the relationship between the initial and final covariance matrices of a generic n-mode Gaussian state in two special but important noisy models. It allows us to conveniently calculate the evolved covariance matrix of any Gaussian state without the need to solve the master equation of the reduce density operator associated with the system of interest. Moreover, an example is given to illustrate the effectiveness of our results. Of course, the interaction between quantum system and its surroundings is not easy to understand. Our results are shown to be valid in the case of noisy Gaussian channel. Other noisy models have not yet been investigated. However, it is hoped that the present research has shed some light upon the subject.

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