CHIN. PHYS. LETT. Vol. 26, No. 2 (2009) 020302

An Alternative Method for Calculating Bound-State of Energy Eigenvalues of Klein–Gordon for Quasi-exactly Solvable Potentials *

Eser Olğar**

Department of Physics and Engineering, University of Gaziantep, Gaziantep, Turkey

(Received 11 September 2008)

We obtain the bound-state energy of the Klein–Gordon equation for some examples of quasi-exactly solvable potentials within the framework of asymptotic iteration method (AIM). The eigenvalues are calculated for type-1 solutions. The whole quasi-exactly solvable potentials are generated from the defined relation between the vector and scalar potentials.

PACS: 03.65.Ge; 03.65.Fd

The Klein–Gordon (KG) equation plays an important role in relativistic quantum mechanics to describe the relativistic effects. In literature, there have been a huge number of studies on the KG equation with mixed exactly solvable potentials by considering different methods.[1−12] In most of these studies, the scalar ($S(x)$) and vector ($V(x)$) potentials have been considered equal to each other to tackle the problem. In this Letter, to conform the condition of having bound state solution ($S(x) \geq V(x)$) we take the general transformation described in our previous paper.[13] In Ref. [13], the KG equation was solved for the exactly solvable potential case. Here the procedure is applied for the quasi-exactly solvable (QES) case.

Recently, an alternative method, the asymptotic iteration method (AIM) is introduced[14] and applied to solve the exactly solvable potentials in the Schrödinger equation.[14−22] Due to more applications to the Schrödinger equation for many potentials, the number of studies on KG equation with AIM is few. AIM has been used to obtain the whole spectra of KG equation for generalized Hulthen potential,[23] Rosen-Morse-type potentials,[24] the Kratzer potential in d dimensions,[25] exponential scalar and vector potentials,[26] and Morse potentials.[13]

Beside the exactly-solvable systems, the QES systems have been attracted great interest.[27−34] In these studies, QES problems have been extensively studied in an algebraic approach. In this study, we try to solve the KG equation for quasi-exactly solvable potentials[27] within the framework AIM. In detail, after introducing the relationship between vector and scalar potentials, we investigate the calculation of the bound state energy eigenvalues of quasi-exactly solvable potentials for the s-wave Klein–Gordon equation within the framework of the asymptotic iteration method (AIM). The general concept of the AIM is outlined. After calculating the appropriate vector potential for obtaining quasi-exactly solvable potentials, AIM is applied to obtain bound state energy eigenvalues for class-I and class-IX potentials. The energy values are tabulated.

The s-wave Klein–Gordon equation with mixed scalar potential $S(x)$ and vector potential $V(x)$ can be written as[8] ($\hbar = 1, c = 1$)

$$\left\{\frac{d^2}{dx^2} + [E - V(x)]^2 - [m + S(x)]^2\right\} f(x) = 0, \quad (1)$$

where E is the energy, and m is the mass of the particle and the radial wavefunction is expressed as $R(x) = f(x)/x$. In order to have a real bound state solutions, the relation between the potentials should be $S(x) \geq V(x)$. Under the condition of equally scalar and vector potential, Eq. (1) becomes the well known Schrödinger-like equation. However, in this work we take into account the case of $S(x) > V(x)$ for the KG equation. Equation (1) is reduced after some arrangement to

$$\left\{\frac{d^2}{dx^2} - [V_{\text{eff}}(x) - (E^2 - m^2)]\right\} f(x) = 0, \quad (2)$$

where

$$V_{\text{eff}} = [S^2(x) - V^2(x)] + 2[mS(x) + EV(x)].$$

The relationship between the corresponding potentials was considered to be[35]

$$V(x) = V_0 + \beta S(x),$$

where V_0 and β are arbitrary constants. In this constraint, we must take the parameter β less then 1, if $V_0 = 0$ for reaching the bound state condition. To avoid this restriction, in Ref. [13], we have defined

$$S(x) = V(x)(\beta - 1), \quad \beta \geq 0. \quad (3)$$

Substituting this expression in the s-wave KG equation yields

$$\left\{\frac{d^2}{dx^2} - [V(x)(V(x)A^2 + B) - \varepsilon]\right\} f(x) = 0, \quad (4)$$

*Supported by the Research Fund of Gaziantep University and the Scientific and Technological Research Council of Turkey (TÜBİTAK).

**Email: olgar@gantep.edu.tr

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where $\varepsilon = (E^2 - m^2)$, $A^2 = (\beta^2 - 2\beta)$ and $B = 2[E - m(1-\beta)]$. After some algebraic calculation, the s-wave KG can be written as

$$\left\{\frac{d^2}{dx^2} - \left[\left[AV(x) + \frac{B}{2A}\right]^2 - \delta\right]\right\} f(x) = 0, \tag{5}$$

where $\delta = \varepsilon + (B/2A)^2$.

Now, after this formalism of the KG equation for unequal vector and scalar potentials, the concept of the AIM that can be used to obtain the spectra of quasi-exactly solvable potentials in KG is shown in the following.

The AIM[14] is proposed and applied to the differential equations in the form

$$y''(x) = \lambda_0 y'(x) + s_0 y(x), \tag{6}$$

where λ_0 and s_0 are functions and $y'(x)$ and $y''(x)$ denotes derivative of y with respect to x. The $(n+2)$th derivative of the function $y(x)$ can be written as

$$y^{(n+2)}(x) = \lambda_n y'(x) + s_n y(x), \tag{7}$$

where λ_n and s_n parameters are given by the recurrence relations

$$\begin{aligned} \lambda_n &= \lambda'_{n-1} + s_{n-1} + \lambda_{n-1}\lambda_0, \\ s_n &= s'_{n-1} + \lambda_{n-1} s_0. \end{aligned} \tag{8}$$

For sufficiently large n,

$$\frac{\lambda_n}{s_n} = \frac{\lambda_{n-1}}{s_{n-1}} = \alpha(x). \tag{9}$$

After introducing these definitions, then the solution of Eq. (6)[14] is

$$\begin{aligned} y(x) = \exp\Big(-\int^x \alpha dt\Big)\Big[C_1 \\ + C_2 \int^x \exp\Big(\int^s (\lambda_0 + 2\alpha) dt\Big) ds\Big]. \end{aligned}$$

The parameters λ_n and s_n in Eq. (8) is calculated for $n = 0$ by taking the initial conditions as $\lambda_{-1} = 1$ and $s_{-1} = 0$ [16] and $\Delta_n(x) = 0$. That is,

$$\Delta_n(x) = \lambda_n(x) s_{n-1}(x) - \lambda_{n-1}(x) s_n(x), \tag{10}$$

where $\Delta_n(x)$ is the termination condition method in Eq. (9).

We shall now obtain of Eq. (2) for different choices of $S(x)$ and $V(x)$. The potentials can be derived from the vector and scalar potentials as listed in Table 1. We consider the exponential-type potentials and inversely polynomial-type potentials, respectively.

In the following, we apply AIM to the two different potential families.

In the exponential potential family, we consider the class-1 potential form as in the Turbiner classification. The corresponding vector potential is

$$V(r) = a\exp(-\alpha x) + b\exp(\alpha x).$$

In this case, the eigenvalue problem in Eq. (5) reduces to

$$\left\{\frac{d^2}{dx^2} - [V_{KG}(x) + \varsigma]\right\} f(x) = 0, \tag{11}$$

where

$$\begin{aligned} V_{KG}(x) &= A^2 b^2 \exp(-2\alpha x) + bB\exp(-\alpha x) \\ &\quad + aB\exp(\alpha x) + a^2 A^2 \exp(2\alpha x), \\ \varsigma &= 2abA^2 + \frac{B^2}{4A^2} - \delta. \end{aligned}$$

After redefining the variable as $z = \exp(\alpha x)$ and using $\psi = z^{-1/2}\chi$ in Eq. (11), we reach the following differential equation

$$\begin{aligned} \left[-\frac{d^2}{dz^2} + \frac{A_1^2}{z^4} + A_2^2 - \frac{A_3}{z^3} - \frac{A_4}{z} + \frac{\gamma(\gamma+1)}{z^2}\right] \\ \cdot \chi(z) = 0, \end{aligned} \tag{12}$$

Table 1. Vector potentials for generating quasi-exactly solvable potentials.

Class	Vector potential $V(r)$	KG Potential
1	$a\exp(-\alpha x) + b\exp(\alpha x)$	$-A^2b^2\exp(-2\alpha x) - bB\exp(-\alpha x) - aB\exp(\alpha x)$ $-a^2A^2\exp(2\alpha x) - 2abA^2 - \frac{B^2}{4A^2}$
2	$a\exp(-2\alpha x) + b\exp(-\alpha x)$	$-A^2(a^2\exp(-4\alpha x) + 2a\exp(-3\alpha x)) - (A^2b^2$ $+aB)\exp(-2\alpha x) - Bb\exp(-\alpha x) - \frac{B^2}{4A^2}$
3	$a\exp(2\alpha x) + b\exp(\alpha x)$	$-A^2(a^2\exp(4\alpha x) + 2a\exp(3\alpha x)) - (A^2b^2 + aB)$ $\cdot\exp(2\alpha x) - Bb\exp(\alpha x) - \frac{B^2}{4A^2}$
4	$\sqrt{a\cos(\alpha x)^4 + b\cosh(\alpha x)^2 + c\cosh(\alpha x)^{-2}} - \frac{B}{2A^2}$	$aA^2\cos(\alpha x)^4 + bA^2\cosh(\alpha x)^2 + cA^2\cosh(\alpha x)^{-2}$
5	$a\cosh(\alpha x)^{-3} + b\cosh(\alpha x)^{-1} - \frac{B}{2A^2}$	$-a^2A^2\cosh(\alpha x)^{-6} - 2aA^2b\cosh(\alpha x)^{-4} - A^2b^2\cosh(\alpha x)^{-2}$
6	$ax^3 + bx - \frac{B}{2A^2}$	$-a^2A^2x^6 - 2aA^2bx^4 - A^2b^2x^2$
7	$ax^3 + bx + \frac{c}{x} - \frac{B}{2A^2}$	$-a^2A^2x^6 - 2aA^2bx^4 - (2aA^2c + b^2A^2)x^2 - \frac{A^2c^2}{x^2} - 2A^2bc$
8	$\frac{a}{x} + bx$	$-\frac{a^2A^2}{x^2} - \frac{aB}{x} - bBx - b^2A^2x^2 - 2aA^2 - \frac{B^2}{4A^2}$
9	$\frac{a}{x} + \frac{b}{x^2}$	$-\frac{b^2A^2}{x^4} - \frac{2abA^2}{x^3} - \frac{(2a^2A^2+bB)}{x^2} - \frac{aB}{x} - \frac{B^2}{4A^2}$
10	$a\cos(\alpha x)$	$-A^2a^2\cos^2(\alpha x) - aB\cos(\alpha x) - \frac{B^2}{4A^2}$

where $A_1 = \dfrac{Aa}{\alpha}$, $A_2 = \dfrac{Ab}{\alpha}$, $A_3 = -\dfrac{aB}{\alpha^2}$, $A_4 = -\dfrac{bB}{\alpha^2}$ and $\gamma(\gamma+1) = \dfrac{(\alpha^2 - 4\varsigma)}{4\alpha^2}$. After obtaining the asymptotic form of $\chi(z)$, the equation becomes amenable to AIM. In order to obtain the asymptotic form of the wave function, we set $z \to 0$ and $z \to \infty$. We reach the asymptotic form

$$\chi(z) \sim z^{s+1} \exp\Big(-\frac{A_1}{z} - A_2 z\Big).$$

Subsequently, after substituting $\chi(z) = z^{s+1}\exp\big(-\frac{A_1}{z} - A_2 z\big) f(z)$ into Eq. (12), we obtain

$$f'' = 2\Big(A_2 - \frac{s+1}{z} - \frac{A_1}{z^2}\Big) f' + \Big(\frac{u}{z} - \frac{v}{z^3}\Big) f, \quad (13)$$

where $u = 2(s+1)A_2 - A_4$, $v = 2sA_1 + A_3$ and $s(s+1) = \gamma(\gamma+1) + 2A_1A_2$. When we compare Eq. (13) with Eq. (6), λ_0 and s_0 are obtained to be

$$\lambda_0 = 2\Big(A_2 - \frac{s+1}{z} - \frac{A_1}{z^2}\Big), \quad s_0 = \frac{u}{z} - \frac{2sA_1 + A_3}{z^3}.$$

Table 2. Energy values for type-1 solution when $a = b = \alpha = m = 1$, and $\beta = 2$.

n	$u = -2nA_2$	$v = -2A_1$	s	E (class I)	E (class IX)
0	0	−2	−1	−0.866025	infinity
			−1	0.866025	infinity
1	−2	−2	−2.0	−1.11803	infinity
			−2.0	1.11803	infinity
			−0.5	−0.829156	−1.00000
			−0.5	0.829156	0.384615
2	−4	−2	−2.63571	−1.35197	−1.000000
			−2.63571	1.35197	−0.765681
			−1.71698	−1.02847	−1.000000
			−1.71698	1.02847	0.244182
			−0.14731	−0.847701	−1.000000
			−0.14731	0.847701	0.78113
3	−6	−2	−3.12512	−1.55252	−1.000000
			−3.12512	1.55252	−0.132878
			−2.50000	−1.29904	−1.000000
			−2.50000	1.29904	0.384615
			−1.51563	−0.972305	−1.000000
			−1.51563	0.972305	0.721141
			0.140750	−0.888898	−1.000000
			0.140750	0.888898	0.889782

If we apply the AIM quantization condition, that is, from Eq. (4) by substituting λ_0 and s_0, in order to obtain polynomial solutions we can find that the parameters must be equal to $u = -2nA_2$ and $v = -2A_1$. The AIM quantization condition gives two types of solution sets for polynomial solutions, types 1 and type 2. These are, for example, for $n = 2$, $v = -2A_1$ (as given above) and $v = \frac{6}{A_2}\big((s+2)(2s+3) - A_1A_2\big)$, respectively. If we use the first value of v we obtain the results as listed in Table 2. However, one can use the second value of s to obtain different results which give us the polynomials. We will not give the second situation here because of complication on the calculations. In this study, we use only $v = -2A_1$ type solution. If we use $u = 2(s+1)A_2 - A_4$, $v = 2sA_1 + A_3$, $s(s+1) = \gamma(\gamma+1) + 2A_1A_2$, $u = -2nA_2$ and $v = -2A_1$, we have $A_4 = 2(n+s+1)A_2$ and $A_3 = -2(s+1)A_1$. If we go back to definitions of A_1, A_2, A_3 and A_4 we find the following expression for the energy as follows:

$$\varsigma = \frac{1}{4}\alpha^2\Big(1 - s(s+1) - \frac{C(E,n)}{\alpha^2}\Big),$$

where $C(E,n) = \dfrac{ab(E + m(\beta-1))\sqrt{(\beta-2)\beta}}{(n+s+2)\alpha^3}$. Then we obtain

$$E^2 - m^2 = \frac{1}{4}\Big(s(s+1) - 1)\alpha^2 + 2ab(\beta-2)\beta + \frac{C(E,n)}{2}\Big).$$

The energy values are listed in Table 2 for both the potentials of type 1 by considering the parameters $a = b = \alpha = m = 1$ and $\beta = 2$. The β value is chosen to be 2 for obtaining non-complex energies for simplicity. Similarly, in Table 3, we present the results for different values of s with $A_1 = A_2 = 1$ for type-2 solution.

Table 3. Different values of s for type-2 solution when $A_1 = A_2 = 1$.

n	$u = -2nA_2$	v	s
0	0	−2	−1.00000
1	−2	−2	−2.00000
			−0.50000
2	−4	$6(s+2)(2s+3)$	−2.50000
			−2.22871
			−2.00000
			−1.27129
			−1.0
3	−6	$\dfrac{4(12s^3 + 72s^2 + 125s + 54)}{10s + 23}$	3.13571
			−2.80396
			−2.63188
			−2.50000
			−2.21698
			−2.11304
			−1.86812
			−1.08301
			−0.647315

In the family of the inversely polynomial-type potentials, we deal with the class-IX potential whose vector potential is obtained to be

$$V(x) = \frac{a}{x} + \frac{b}{x^2}.$$

After replacing x by z, the eigenvalue problem reduces to

$$\Big[-\frac{d^2}{dz^2} + \frac{A_1^2}{z^4} + A_2^2 - \frac{A_3}{z^3} - \frac{A_4}{z} + \frac{\gamma(\gamma+1)}{z^2}\Big] \cdot \chi(z) = 0, \quad (14)$$

where $A_1 = Ab$, $A_2 = \sqrt{\frac{B^2}{4A^2} - \delta}$, $A_3 = -2aA^2b$, $A_4 = -aB$ and $\gamma(\gamma+1) = a^2A^2 + bB$. This inversely polynomial potential is the class-IX potential form. By applying the same procedure performed in the previous potential form, when $z \to 0$ and $z \to \infty$, $\chi(z) \sim z^{s+1}\exp\big(-\frac{A_1}{z} - A_2 z\big)$, and after substituting

$\chi(z) = z^{s+1}\exp\big(-\frac{A_1}{z} - A_2 z\big)f(z)$ into Eq. (14), we have

$$f'' = 2\Big(A_2 - \frac{s+1}{z} - \frac{A_1}{z^2}\Big)f' + \Big(\frac{u}{z} - \frac{v}{z^3}\Big)f,$$

where $u = 2(s+1)A_2 - A_4$, $v = 2sA_1 + A_3$ and $s(s+1) = \gamma(\gamma+1) + 2A_1A_2$. In this form of differential equation, if we apply AIM to this final equation with $\lambda_0 = 2\big(A_2 - \frac{s+1}{z} - \frac{A_1}{z^2}\big)$ and $s_0 = \frac{u}{z} - \frac{2sA_1+A_3}{z^3}$. Then, we obtain the exact solutions of Eq. (14) under some constrains on the parameters A_1, A_2, A_3 and A_4. It can be easily seen that the solutions of this form of potential is equal to the solution of the first example except for definitions of A_1, A_2, A_3 and A_4. In this type of potential, the definitions of energy for the type-1 solution can be obtained to be

$$E^2 - m^2 = -\left(\frac{a(E + m(\beta-1))}{n+s+1}\right)^2. \qquad (15)$$

Similar to the class-I potential, the corresponding values obtained for the second type solution in Table 3 are valid for this potential. For simplicity, the energy values for $k = 1$ are also tabulated in Table 2.

Similar to the procedure presented here, it is possible to obtain the same solutions for other class potentials described by Turbiner.[27] All the potential classes satisfy the requirements of using the AIM, which is conditions of the differential equations being transformable to a specific form in Eq. (2), with quite restrictive asymptotic conditions on the coefficient functions of $\alpha(x)$ in Eq. (5).

We have obtained the energy values of quasi-exactly solvable exponential and inversely polynomial potentials for KG equation within the framework AIM method. We have shown that there are appropriate vector and scalar potentials in order to generate the all quasi-exactly solvable potentials in the extent of general transformation defined for the corresponding potentials. It is clear that AIM is an alternative method for obtaining the energy values of QES potentials.

The author is grateful to Ramazan Koç for useful discussions.

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