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Ground State Energy for Fermions in a 1D Harmonic Trap with Delta Function Interaction *

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Conjectures are made for the ground state energy of a large spin 1/2 Fermion system trapped in a 1D harmonic trap with delta function interaction. States with different spin J are separately studied. The Thomas–Fermi method is used as an effective test for the conjecture.

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There is recent experimental and theoretical interest[1,2] in a model Hamiltonian for N one-dimensional spin 1/2 Fermions in a harmonic trap:

$$H=\sum_{i=0}^{N}\left[-\frac{1}{2}\frac{\partial^2}{\partial x_i^2}+\frac{1}{2}x_i^2\right]+g\sum_{i>j}\delta(x_i-x_j). \quad (1)$$

For $N=2$, the eigenvalue problem of this Hamiltonian had been analytically solved.[3] We concentrate in the present paper on the ground state energy E_J for a given total spin J as $N\to\infty$. It is well known from group theory that for spin J the spin wave function is described by the two-row Young pattern $[N-M,M]$ where $M=(N-2J)/2$ and the space wave function by its associated pattern.[4] According to a theorem due to Lieb and Mattis,[5] when g is finite,

$$E_J<E_{J'}, \quad \text{if} \quad J<J'.$$

We notice the following facts:
(A) As $g\to+\infty$, the ground state energy E_J for any total spin J approaches[2] a limit

$$E_J\to\sum_{n=0}^{N-1}\left(\frac{1}{2}+n\right)=\frac{1}{2}N^2. \quad (2)$$

(B) For $g=0$, the ground state wave function for spin $J=N/2-M$ can be taken to be a product of two determinants:[4]

$$\Psi=\det\left[u_0(x_1)u_1(x_2)\cdots u_{N-M-1}(x_{N-M})\right]\times\det\left[u_0(x_{N-M+1})\cdots u_{M-1}(x_N)\right], \quad (3)$$

where $u_i(x)$ is the normalized eigenfunction of the harmonic oscillator. Its energy is

$$E_J=\frac{1}{2}\left[(N-M)^2+M^2\right]. \quad (4)$$

When $J=0$, $N=2M$ and $E_0=N^2/4$. Equations (2) and (4) indicate that

$$\frac{1}{4}\le\frac{E_0}{N^2}\le\frac{1}{2}, \quad \text{for} \quad g\ge 0. \quad (5)$$

(C) For $g\to-\infty$, when the total spin $J=0$, $M=N/2$ pairs are formed with spatial size of the order of $-g^{-1}$ each. The internal energy of each pair is $-g^2/4$. Thus the M pairs contribute $-Mg^2/4$ to the total energy. Between these pairs there are Fermionic repulsion as well as attractive delta function interaction. It is difficult to disentangle this complicated repulsion-attraction mix. However we observe that the Fermionic repulsion in the absence of attractive g is given by (5). Thus it is reasonable to surmise that as $g\to-\infty$, the repulsion contributes X to E_0:

$$E_0\to-\frac{g^2}{4}M+X, \quad \text{as} \quad g\to-\infty, \quad (6)$$

where $N^2/2\ge X\ge 0$. The X term in (6) is small compared with the other term as $g\to-\infty$. Equation (6) holds for $1\le M\le N/2$.

Equation (5) shows that for $g\ge 0$, E_0 is of order N^2 while (6) shows that E_0 is of order M as $g\to-\infty$. *How can that be?* This question leads to the following conjecture:

Conjecture 1: As $N=2M$ and $N\to\infty$, the E_0/N^2 versus $t=g/\sqrt{N}$ curve approaches a limit:

$$\frac{E_0}{N^2}\to f_0\Big(\frac{g}{\sqrt{N}}\Big). \quad (7)$$

(2), (4) and (6) show that

$$f_0(t)\to 1/2, \quad \text{as} \quad t\to+\infty, \quad (8)$$
$$f_0(0)=1/4, \quad (9)$$
$$f_0(t)\to -t^2/8, \quad \text{as} \quad t\to-\infty. \quad (10)$$

The schematic figure of conjecture is shown in Fig. 1 and its generalization is as follows.

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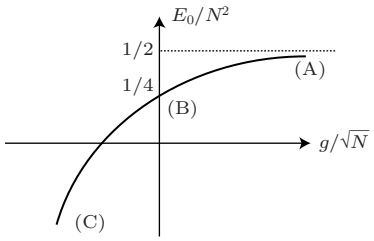


Fig. 1. Ground state energy E_J/N^2 versus $g/\sqrt{N}$ curve for $J = 0$ (Schematic). Equations (8) (9) and (10) refer respectively to regions A, B and C in this figure.

Conjecture 2: For fixed $J/N = b$, the E_J/N^2 versus $t = g/\sqrt{N}$ curve approaches a limit

$$\frac{E_J}{N^2} \to f_b\Big(\frac{g}{\sqrt{N}}\Big), \tag{11}$$

where

$$f_b(t) \to 1/2, \quad \text{as} \quad t \to +\infty, \tag{12}$$
$$f_b(0) = 1/4 + b^2, \tag{13}$$
$$f_b(t) \to -\frac{1}{4}\Big(\frac{1}{2} - b\Big)t^2, \text{ as } t \to -\infty. \tag{14}$$

Simple Test of the Two Conjectures. It seems difficult to prove Conjectures 1 and 2. However, there is a very simple but effective test: If Conjectures 1 and 2 are correct, then for large N and given J, the slope at $g = 0$ is

$$\frac{dE_J}{dg} = N^{3/2}\frac{d(E_0/N^2)}{d(g/\sqrt{N})} \to N^{3/2}\frac{df_b(t)}{dt}, \tag{15}$$

where $t = g/\sqrt{N}$. Namely, the slope should be proportional to $N^{3/2}$ for large N.

We calculate dE_J/dg at $g = 0$ by first order perturbation theory,

$$\frac{dE_J}{dg} = \left\langle \left| \int \psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x)\psi_\downarrow(x)\psi_\uparrow(x)dx \right| \right\rangle, \tag{16}$$

where $\psi_\uparrow$ and $\psi_\downarrow$ are fermionic annihilation operators and the bra & ket designate the ground state at $g = 0$. According to (3) this ground state has $N - M$ spin up particles and M spin down particles in single particle states u_0, u_1, ... Thus we have

$$\frac{dE_J}{dg} = \sum_{i=0}^{N-M-1}\sum_{j=0}^{M-1}\int u_i^2(x)u_j^2(x)dx.$$

Defining

$$S[N-M, M] = N^{-3/2}\frac{dE_J}{dg}\Big|_{g=0}, \tag{17}$$

we have

$$S[N-M, M] = N^{-3/2}\int \rho_{N-M}(x)\rho_M(x)dx, \tag{18}$$

where

$$\rho_M(x) = \sum_{i=0}^{M-1} u_i^2(x). \tag{19}$$

The physical meaning of $\rho_M(x)$ is the density of the up-spin (or down-spin) particles at $g = 0$.

The density $\rho_M(x)$ and the slope $S[N-M, M]$ can be calculated in a straightforward way with a computer. We define a scaled density function:

$$R_M(y) = \frac{1}{\sqrt{2M}}\rho_M(\sqrt{2M}y). \tag{20}$$

From (19) we have, as M approaches infinity,

$$R_M(y) \to 0, \qquad |y| \geq 1. \tag{21}$$

Figure 2 shows that $R_M(y)$ tends to a limit as $M \to \infty$ which we shall calculate by the Thomas–Fermi method later in this paper. Here we list in Table 1 values of the slope $S[N-M, M]$ for some given parameter $b = J/N$. One can see from the table that for each value of parameter b, $S[N-M, M]$ rapidly converges to a limit as N increases, *supporting both conjectures 1 and 2.*

Thomas-Fermi Method For large values of M, we can evaluate ρ_M by the Thomas–Fermi method (cf. Fig. 3). When M is very large, the particles in dx have as their maximum kinetic energy (= maximum total energy minus the potential) $\sim M - x^2/2$. Thus, the number of particles in dx is $\sqrt{2M - x^2}dx/\pi$. I.e.,

$$\rho_M(x) \to \frac{1}{\pi}\sqrt{2M - x^2}, \quad \text{as} \quad M \to \infty. \tag{22}$$

Thus from (20) we have as $M \to \infty$

$$R_M(y) \to \sqrt{1-y^2}/\pi, \quad \text{for} \quad |y| \leq 1. \tag{23}$$

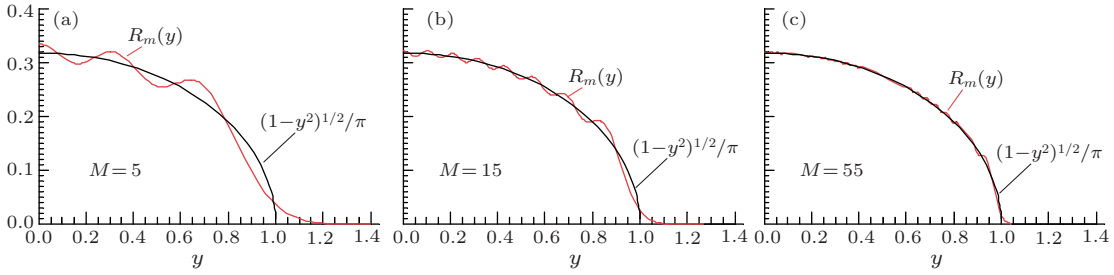


Fig. 2. Comparison of $R_M(y)$ with its limit $\sqrt{1-y^2}/\pi$.

Table 1. The slope $S[N-M,M]$ for different b.

$b=0$	$S[N-M,M]$	$b=1/6$	$S[N-M,M]$
$[1,1]$	0.141047	$[2,1]$	0.115165
$[3,3]$	0.136147	$[4,2]$	0.114516
$[5,5]$	0.135539	$[6,3]$	0.114389
$[8,8]$	0.135292	$[8,4]$	0.114343
$[10,10]$	0.135228	$[10,5]$	0.114322
$[12,12]$	0.135191	$[12,6]$	0.114311
$[15,15]$	0.135160	$[14,7]$	0.114304
$[18,18]$	0.135135	$[16,8]$	0.114299
$[19,19]$	0.135105	$[18,9]$	0.114296
$[20,20]$	0.135097	$[20,10]$	0.114294
Eq. (24)	0.135095	Eq. (24)	0.114284
$b=1/4$	$S[N-M,M]$	$b=1/10$	$S[N-M,M]$
$[3,1]$	0.0935021	$[3,2]$	0.127119
$[6,2]$	0.0932855	$[6,4]$	0.126726
$[9,3]$	0.0932444	$[9,6]$	0.126651
$[12,4]$	0.0932299	$[12,8]$	0.126624
$[15,5]$	0.0932232	$[15,10]$	0.126611
$[18,6]$	0.0932195	$[18,12]$	0.126604
Eq. (24)	0.0932112	Eq. (24)	0.126589
$b=3/10$	$S[N-M,M]$	$b=1/14$	$S[N-M,M]$
$[4,1]$	0.0780554	$[4,3]$	0.130760
$[8,2]$	0.0779578	$[8,6]$	0.130501
$[12,3]$	0.0779394	$[12,9]$	0.130451
$[16,4]$	0.0779330	$[16,12]$	0.130434
$[20,5]$	0.0779300	$[20,15]$	0.130429
Eq. (24)	0.0779247	Eq. (24)	0.130411

Considering (20)–(23), we obtain the slope $S[N-m,M]$ as $N\to\infty$ from (18)[7]

$$\lim_{N\to\infty} S[N-M,M] = \frac{2\sqrt{1+2b}}{3\pi}\left[F\left(-\frac{1}{2},\frac{1}{2};1,z\right) - 2bF\left(\frac{1}{2},\frac{1}{2};1,z\right)\right], \qquad (24)$$

where $b=J/N$, $z=(1-2b)/(1+2b)$, and F is the hypergeometric function. For $J=0$, (24) reduces to

$$\lim_{N\to\infty} S[M,M] = \frac{4}{3\pi^2} \sim 0.135095. \qquad (25)$$

The numerical values of (24) are listed in the last line for each value b in Table 1 for comparison.

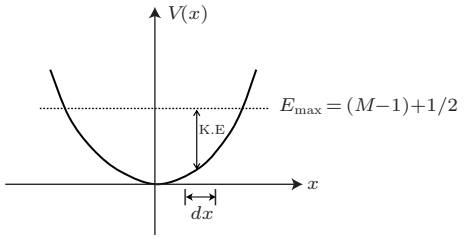


Fig. 3. Thomas-Fermi Method.

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References

[1] Olshanii M 1998 *Phys. Rev. Lett.* **81** 938
Bloch I et al 2008 *Rev. Mod. Phys.* **80** 885
[2] Guan L M et al 2009 *Phys. Rev. Lett.* **102** 160402
[3] Aneziris C, Balachandran A P and Sen D 1991 *Inter. J. Mod. Phys.* A **6** 4721
Busch T et al 1998 *Found. Phys.* **28** 549
[4] Yang C N 2009 *Chin. Phys. Lett.* **26** 120504. We will follow the notation of this paper.
[5] Lieb E and Mattis D 1962 *Phys. Rev.* **125** 164
[6] Girardeau M D and Minguzzi A 2007 *Phys. Rev. Lett.* **99** 230402
[7] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series, and Products* (New York: Academic)