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A Model for Fractal Dimension of Rough Surfaces *

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We report a model for the fractal dimension D_s of rough surfaces based on the fractal distribution of roughness elements on surfaces and the fractal character of surface profiles. The proposed model for the fractal dimension D_s is expressed as a function of the fractal dimensions D for conic roughness diameter/height and D_p for surface profile, maximum roughness base diameter $\lambda_{\max}$, the ratio β of conic roughness height to its base radius as well as the ratio $\lambda_{\min}/\lambda_{\max}$ of the minimum to the maximal base diameter.

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Characterization of surface morphology (surface roughness and topography) is of great interest[1−10] in sciences and technologies because transport properties in materials and many engineering problems are often significantly influenced by the surface morphology. Conventional methods such as scanning electron microscopy (SEM) and atomic force microscopy (AFM) for description of surface morphology can provide detailed morphological information. However, they cannot provide quantitative morphological description.[10] In the description of surface morphology, the standardized parameters such as the average roughness, the rms surface roughness and distribution function are used for the quantitative characterization of surfaces. However, they cannot be used to describe the irregularity or complexity of surfaces. Fractal geometry is often used to describe highly disordered systems, which are characterized by their invariability at any scale used to represent them. In description of fractal surfaces, the fractal dimensions for surfaces and for profiles are often used to characterize the degree of roughness or irregularity of a surface. The degree of irregularity of a surface may be described by the fractal dimension D_s. It is commonly recognized that the fractal dimension D_p for surface profiles varies from 1.0 (a straight line) to the maximum value of 2.0 (representing a highly tortuous profile that fills a plane). The fractal dimension D_s for surface varies from 2.0 (perfectly smooth surface) to the maximum value of 3.0 (representing an extremely rough surface that fills a three-dimensional space). The relationship between the surface fractal dimension D_s and the fractal dimension D_p for surface profiles was thought to be related by $D_s = D_p + 1$ for isotropic rough surfaces.[7,11,12] However, up to date, no quantitative justification has been reported.

In this Letter, we report a model for the fractal dimension D_s for rough surfaces based on the fractal distribution of roughness elements on surfaces and the fractal character of surface profiles.

Fractal objects are usually defined to be independent of scales of measurement and to follow the scaling law in the form of[11,13]

$$M(L) \sim L^D, \tag{1}$$

where M can be the length of a line or the area of a surface or the volume of an object, and D is the fractal dimension. Equation (1) implies the property of self-similarity, which means that the value of D from Eq. (1) keeps constant over a range of length scales, L. The geometry structures such as the Sierpinski gasket, Sierpinski carpet and Koch curve are the examples of the exactly self-similar fractals, which exhibit the self-similarity over an infinite range of length scales. However, exactly self-similar fractals in a global sense are rarely found in nature. Numerous objects found in nature are not exactly self-similar, and they are statistically self-similar. Statistical self-similarity implies that these objects exhibit the self-similarity in some average sense and over a certain range of length scales, L.

Fractal objects are also defined as that their cumulative size-distributions (such as contact spots on engineering surfaces and pores in porous media) follow the following scaling law:[1,2,11,14,15]

$$N(L \geq \lambda) = \left(\frac{\lambda_{\max}}{\lambda}\right)^D. \tag{2}$$

Equation (2) is valid for the description of fractal objects if the exponent D remains constant in a statistically self-similar region with $0 < D < 2$ in two dimensions and $0 < D < 3$ in three dimensions. In Eq. (2), λ is the diameter of an island/spot/pore and $\lambda_{\max}$ the

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maximum diameter. Equation (2) indicates that the cumulative size-distribution of fractal objects whose sizes are greater than or equal to λ follows the fractal scaling law in the range of $\lambda_{\min} - \lambda_{\max}$. The equality in Eq. (2) implies that there is only one maximum island on the Earth or one maximum spot on a surface or one maximum pore in a set of fractal pores.

Differentiating Eq. (2) with respect to λ results in the number of objects whose sizes are within the infinitesimal range from λ to $\lambda + d\lambda$,

$$-dN = D\lambda_{\max}^{D}\lambda^{-(D+1)}d\lambda. \tag{3}$$

The negative sign in Eq. (3) indicates that the island or particle or spot or pore number decreases with the increase of its size, and $-dN > 0$.

Figure 1 displays a typical configuration of a rough surface, which is assumed to be comprised of a number of randomly distributed cone-like peaks. Rough elements (conic peaks) on rough surfaces have been shown to be self-similar and to be fractals.[6,10] In this work, the following assumptions are made: the peak height distribution is isotropic on a rough surface; the surface profiles may be zigzag shapes, which may be related to conic peak heights/diameters; peaks on a surface do not overlap each other, and therefore the rough surfaces discussed in this work are not self-affine fractals. For self-affine fractal surfaces based on Refs. [2,4], we will derive a new model, which will be addressed elsewhere. Since the cone-like peaks are statistically self-similar, the ratio of the peak height to its base diameter (or radius) is constant, i.e. $h/r = \beta = \text{const}$. Therefore, the fractal dimension D of the size distribution of the approximately circular bases as shown in Fig. 1(c) is the same as that of peak heights, and in this work we attempt to derive a model for the fractal dimension D_s of rough surfaces, which are related to the fractal dimension D for the circular bases/peak heights and the fractal dimension D_p for surface profiles.

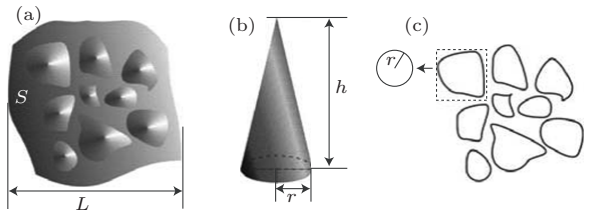


Fig. 1. Typical morphology: (a) top view of a rough surface, (b) a representative cone-like rough element, and (c) the base morphology of (a).

The surface area of a cone is

$$S_1 = \pi r\sqrt{h^2 + r^2} = \frac{\pi}{4}\lambda^2\sqrt{1+\beta^2}, \tag{4}$$

where $r = \lambda/2$ and λ is the diameter of an approximately circular base. In Eq. (4) the base area is not included. Equation (4) is expressed as a function of λ only because β is constant due to self-similarity of conic peaks on a rough surface.

The total surface area of all conic peaks in a unit cell of L can be found by integration from the smallest cone of $\lambda_{\min}$ to the largest cone of $\lambda_{\max}$, i.e.

$$\begin{aligned} S &= \int_{\lambda_{\min}}^{\lambda_{\max}} S_1 \times (-dN) \\ &= \int_{\lambda_{\min}}^{\lambda_{\max}} \frac{\pi}{4}\lambda^2 D\sqrt{1+\beta^2}\lambda_{\max}^{D}\lambda^{-(D+1)}d\lambda \\ &= \frac{\pi\lambda_{\max}^2}{4}\frac{D}{2-D}\sqrt{1+\beta^2}\left[1-\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{2-D}\right]. \end{aligned} \tag{5}$$

Equation (5) shows that the total surface area of all cone-like peaks is a function of the fractal dimension D for base diameters/peak heights, the ratio of the minimum base diameter/peak height to the maximum base diameter/peak height, maximum base diameter $\lambda_{\max}$ and the ratio β of cone height to its base radius.

Wheatcraft and Tyler[17] applied simple scaling arguments to characterize the tortuous streamlines through the heterogeneous media by fractal geometry and developed a fractal scaling for flow through heterogeneous media, and the scaling relationship is given by

$$L_t(\varepsilon) = \varepsilon^{1-D_T}L^{D_T}, \tag{6}$$

where D_T (with $1 < D_T < 2$ in two dimensions), L, L_t and ε are the fractal dimension for tortuous streamlines/streamtubes, the straight distance, the actual length and scale, respectively. The profile of a rough surface is much like the tortuous streamline, and therefore Eq. (6) can also be used to describe the profile of a rough surface. In this situation, the scale ε and D_T are respectively replaced by the cone diameter λ and the fractal dimension D_p for surface profiles, then Eq. (6) is rewritten as

$$L_t(\lambda) = \lambda^{1-D_p}L^{D_p}. \tag{7a}$$

It is clear that if the scale ε or λ is equal to L, the actual length is equal to the straight length or the size of a unit cell. The total length of a profile may be obtained as

$$L_t = \lambda_{\min}^{1-D_p}L^{D_p}. \tag{7b}$$

If a profile is assumed to be formed by a number of cones without overlapping, the total length of a profile can be found by the integral over all cones arranged in a straight line of length L, i.e.

$$L_t = \int L_1(-dN) = \int_{\lambda_{\min}}^{\lambda_{\max}} \sqrt{1+\beta^2}\lambda(-dN), \tag{8}$$

where L_1 is the length of the profile of a cone, $L_1 = 2\sqrt{h^2+r^2} = 2r\sqrt{1+\beta^2} = \lambda\sqrt{1+\beta^2}$, and $-dN$ is given by Eq. (3). After the integration of Eq. (8), we obtain

$$L_t = \frac{D\sqrt{1+\beta^2}}{1-D}\lambda_{\max}\left[1-\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{-D+1}\right]. \tag{9}$$

Equation (9) indicates that the total length of profile of a rough surface depends on several parameters such as D, β, $\lambda_{\max}$ and $\lambda_{\min}/\lambda_{\max}$.

From Eq. (7b), we have

$$L = \left(L_t \lambda_{\min}^{D_p-1}\right)^{1/D_p}, \tag{10}$$

where L_t is determined by Eq. (9). Equations (8) and (10) indicate that when $\beta = 0$, $L_t = L$ and $D_p = 1$, meaning that the surface profile is a straight line and the surface is smooth.

Since rough elements (peaks) on a surface are statistically self-similar, Eq. (1) can also be applied to describe a rough surface. Therefore, applying Eq. (1) to Eq. (5) approximately results in

$$S = L^{D_s}, \tag{11}$$

where D_s is the fractal dimension for rough surfaces and $D_s < 3$.

From Eq. (11) we have

$$L = S^{1/D_s}. \tag{12}$$

For a unit cell of L, let Eq. (10) be equal to Eq. (12), we have

$$\left(L_t \lambda_{\min}^{D_p-1}\right)^{1/D_p} = S^{1/D_s}. \tag{13}$$

From Eq. (13) we have

$$D_s = \frac{D_p \ln S}{\ln(L_t \lambda_{\min}^{D_p-1})}, \tag{14}$$

where S and L_t are determined by Eqs. (5) and (9), respectively. Equation (14) reveals that the fractal dimension for surface quantitatively depends on several parameters: fractal dimension D for cone base diameter or cone height, the maximum base diameter $\lambda_{\max}$, the ratio β of cone height to its base radius and $\lambda_{\min}/\lambda_{\max}$, as well as on the fractal dimension D_p for surface profiles.

In summary, a model for the fractal dimension of rough surfaces is presented based on the fractal distribution of roughness elements on rough surfaces and the fractal character of surface profiles. There is no empirical constant in the proposed model.

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