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Relation between Characteristic Function of Density Operator and Tomogram *

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In terms of the intermediate coordinate-momentum representation (Chin. Phys. Lett. 18 (2001) 850) and using the technique of integration within an ordered product of operators, we put the tomography theory into operator version. We reveal the new relation between the tomogram and the characteristic function of the density operator. The new expansion of the density operator in terms of the intermediate coordinate-momentum representation is also obtained.

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Quantum tomography is about reconstructing the tomogram of quantum states through the Wigner quasidistribution, it is becoming a standard tool for the characterization of the quantum properties of light. In Ref. [1], Vogel and Risken pointed out that the probability distrbution for the rotated quadrature phase can be expressed in terms of the Wigner function,[2] and the reverse is also true (named as the Vogel–Risken relation), i.e., one can obtain the Wigner distribution by the tomographic inversion of a set of measured probability distributions of the quadrature amplitude. Smithey, Beck and Raymer[3] also pointed out that once the distribution is obtained, one can use the inverse Radon transformation familiar in tomographic imaging to obtain the Wigner distribution and density matrix (optical tomographic imaging techniques derive two-dimensional data from a three-dimensional object to obtain a slice image of the internal stucture and thus have the ability to peer inside the object noninvasively). In quantum optics theory, $\mu X + \nu P$ (μ and ν are real numbers, $[X, P] = i\hbar$) represents all possible linear combination of quadratures X and P of the oscillator field mode and can be measured by the homodyne measurement just by varying the phase of the local oscillator. The average of the random outcomes of the measurement, at a given local oscillator phase, is connected with the marginal distribution of the Wigner function, thus the homodyne measurement of the light field permits the reconstruction of the Wigner function of a quantum system by varying the phase of the local oscillator. In Ref. [4] following Dirac's indication:[5] "When one has a particular problem to work out in quantum mechanics, one can minimize the labour by using a representation in which the representatives of the more important abstract quantities occurring in that problem are as simple as possible. we construct a new quantum mechanical representation composing of the eigenvector $|x, \mu, \nu\rangle$ of the operator $\mu X + \nu P$, which is an overcomplete and nonorthogonal representation, and we found that $|x, \mu, \nu\rangle$ is inherent to the Radon transformation (with two parameters) of the Wigner operator. In this Letter, we reveal an important relation between the tomogram and the characteristic function. First, through the Radon transformation of the Wigner operator we naturally introduce the state $|x, \mu, \nu\rangle$, where μ and ν are the two Radon transformation parameters. Then we show that the tomogram is just $\mathrm{Tr}[\rho|x, \mu, \nu\rangle\langle x, \mu, \nu|]$, where ρ is a quantum state (density operator). We demonstrate that the tomogram is just a Fresnel transformation of $\langle x'|\rho|x''\rangle$, where $|x''\rangle$ is the usual coordinate eigenstate. Finally, we reveal an important relation between the tomogram and the characteristic function.

According to the Weyl quantizatiom scheme,[6] the classical correspondence $h(p, x)$ of an operator $H(X, P)$ is given by the relation

$$H(X, P) = \iint dpdxh(p, x)\Delta(p, x), \tag{1}$$

where $\Delta(p, x)$ is the Wigner operator, which in the usual coordinate $|x\rangle$ representation is expressed as

$$\Delta(p, x) = \int_{-\infty}^{+\infty} \frac{d\sigma}{2\pi} e^{ip\sigma} \left| x + \frac{\sigma}{2} \right\rangle \left\langle x - \frac{\sigma}{2} \right|, \tag{2}$$

where

$$|x\rangle = \pi^{-1/4} \exp\left\{ -\frac{x^2}{2} + \sqrt{2}a^\dagger x - \frac{a^{\dagger 2}}{2} \right\} |0\rangle, \tag{3}$$

$|0\rangle$ is ground state annihilated by a, $[a, a^\dagger] = 1$. Using the normally ordered form of the vacuum projector

$$|0\rangle\langle 0| =: e^{-a^\dagger a} : \tag{4}$$

and the technique of integration within an ordered product (IWOP) of operators[7,8] we have derived the

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explicit normally ordered form of Wigner operation[9]

$$\Delta(p,x) = \frac{1}{\pi} : e^{-(p-P)^2-(x-X)^2} : = \frac{1}{\pi} : e^{-2(a^\dagger-\alpha^*)(a-\alpha)} :, \tag{5}$$

where

$$X = \frac{a+a^\dagger}{\sqrt{2}}, \quad P = \frac{a-a^\dagger}{\sqrt{2}i}. \tag{6}$$

When the classical function $h(p,x)$ in Eq. (1) is taken as $\delta(x-\nu p-\mu x)$, the Radon transformation kernel,[10] then Eq. (1) becomes

$$\begin{aligned}&\int_{-\infty}^{+\infty} dp'dx'\ \delta(x-\nu p'-\mu x')\ \Delta(p',x') \\ &= \int_{-\infty}^{+\infty} \frac{dp'dx'}{\pi}\delta(x-\nu p'-\mu x') : e^{-(p'-P)^2-(x'-X)^2} : \\ &= \frac{1}{\sqrt{\pi(\mu^2+\nu^2)}} : \exp\Big[-\frac{1}{\mu^2+\nu^2}[x-(\mu X+\nu P)]^2\Big] : .\end{aligned} \tag{7}$$

Using Eq. (4) we can decompose

$$\begin{aligned}&\frac{1}{\sqrt{\pi(\mu^2+\nu^2)}} : \exp\Big[-\frac{1}{\mu^2+\nu^2}[x-(\mu X+\nu P)]^2\Big] : \\ &= |x,\mu,\nu\rangle\langle x,\mu,\nu|,\end{aligned} \tag{8}$$

where the explicit form of the state vector $|x,\mu,\nu\rangle$ in Fock space is

$$\begin{aligned}|x,\mu,\nu\rangle &= [\pi(\mu^2+\nu^2)]^{-1/4}\exp\Big[-\frac{x^2}{2(\mu^2+\nu^2)} \\ &\quad + \sqrt{2}\frac{x}{-i\nu+\mu}a^\dagger - \frac{i\nu+\mu}{2(-i\nu+\mu)}a^{\dagger 2}\Big]|0\rangle,\end{aligned} \tag{9}$$

so Eq. (7) becomes

$$\begin{aligned}&\int_{-\infty}^{+\infty} dp'dx'\ \delta(x-\nu p'-\mu x')\ \Delta(x',p') \\ &= |x,\mu,\nu\rangle\langle x,\mu,\nu|,\end{aligned} \tag{10}$$

we conclude that the projection operator $|x,\mu,\nu\rangle\langle x,\mu,\nu|$ is just the Radon transformation of the Wigner operator, so the introduction of $|x,\mu,\nu\rangle$ is of necessity. Multiplying both sides of Eq. (10) by a density operator ρ and then performing the trace, noting the Wigner function $W(p',x') = \mathrm{Tr}[\rho\Delta]$, we see

$$\mathfrak{T}(x,\mu,\nu) = \iint dx'dp'\delta(x-\nu p'-\mu x')W(p',x',) \tag{11}$$

where

$$\begin{aligned}\mathfrak{T}(x,\mu,\nu) &\equiv \mathrm{Tr}\big[\rho|x,\mu,\nu\rangle\langle x,\mu,\nu|\big] \\ &= \langle x,\mu,\nu|\rho|x,\mu,\nu\rangle,\end{aligned} \tag{12}$$

is the tomogram of ρ, thus we may name $|x,\mu,\nu\rangle\langle x,\mu,\nu|$ the tomogram operator.

Here we emphasize that the classical Weyl correspondence of $|x,\mu,\nu\rangle\langle x,\mu,\nu|$ is $\delta(x-\mu x'-\nu p')$, which is another view to look at the Randon transformation of Wigner operator.

To our knowledge, in the literature, only the Wigner function's Radon transformation has been discussed, which is due to the fact that with the IWOP technique we can tackle the Radon transformation of the Wigner operator as well as its inverse in the operator version.

It can be proved that the state of $|x,\mu,\nu\rangle$ satisfies the eigenvector equation

$$(\mu X+\nu P)|x,\mu,\nu\rangle = x|x,\mu,\nu\rangle, \tag{13}$$

and $|x,\mu,\nu\rangle$ is complete

$$\begin{aligned}&\int_{-\infty}^{+\infty} dx|x,\mu,\nu\rangle\langle x,\mu,\nu| \\ &= \frac{1}{\sqrt{\pi(\mu^2+\nu^2)}}\int_{-\infty}^{+\infty} dx : \exp\Big\{-\frac{1}{\mu^2+\nu^2} \\ &\quad \cdot\big[x-(\mu\hat{Q}+\nu\hat{P})\big]^2\Big\} := 1.\end{aligned} \tag{14}$$

Thus $|x,\mu,\nu\rangle$ is known as the intermediate coordinate-momentum representation. We now evaluate the overlap $\langle x'|x,\mu,\nu\rangle$ by considering

$$\begin{aligned}\langle x'|(\mu X+\nu P)|x,\mu,\nu\rangle &= \Big(\mu x' - i\nu\frac{\partial}{\partial x'}\Big)\langle x'|x,\mu,\nu\rangle \\ &= x\langle x'|x,\mu,\nu\rangle.\end{aligned} \tag{15}$$

In the following we present the solution

$$\langle x'|x,\mu,\nu\rangle = ce^{ix'(x-\mu x'/2)/\nu}. \tag{16}$$

By taking c a normalization constant independent of x, from Eqs. (14) and (16) we can determine

$$\begin{aligned}\delta(x'-x'') &= \int_{-\infty}^{+\infty} dx\langle x'|x,\mu,\nu\rangle\langle x,\mu,\nu|x''\rangle \\ &= |c|^2\int_{-\infty}^{+\infty}\exp\Big[i\Big(\frac{x'}{\nu}-\frac{x''}{\nu}\Big)x \\ &\quad + i\frac{\mu}{2\nu}\Big(x''^2-x'^2\Big)\Big]dx \\ &= |c|^2\nu 2\pi\delta(x'-x'').\end{aligned} \tag{17}$$

Thus $|c|^2 = \frac{1}{\sqrt{2\pi\nu}}$ and

$$\langle x'|x,\mu,\nu\rangle = \frac{1}{\sqrt{2\pi\nu}}\exp\Big[\frac{ixx'}{\nu}-\frac{i\mu}{2\nu}x'^2\Big]. \tag{18}$$

It follows

$$|x,\mu,\nu\rangle = \frac{1}{\sqrt{2\pi\nu}}\int_{-\infty}^{+\infty} dx\exp\Big[\frac{ixx'}{\nu}-\frac{i\mu}{2\nu}x'^2\Big]|x'\rangle, \tag{19}$$

and

$$\mathfrak{T}(x,\mu,\nu) = \frac{1}{2\pi\nu}\iint dx''dx'\langle x'|\rho|x''\rangle$$

$$\cdot \exp\left[\frac{ix}{\nu}(x'-x'')+\frac{i\mu}{2\nu}(x''^2-x'^2)\right], \tag{20}$$

which is a kind of Fresnel transformation. When $\rho = |\psi\rangle\langle\psi|$, a pure state, then the tomogram $\mathfrak{T}(x,\mu,\nu) = |\langle x,\mu,\nu|\psi\rangle|^2$.

Let $x' = \dfrac{\alpha+\alpha^*}{\sqrt{2}}$, $p' = \dfrac{\alpha-\alpha^*}{\sqrt{2}i}$, consequently, $\Delta(p',x')$ is denoted as $\Delta(\alpha)$, from Eq. (5) we see

$$\begin{aligned}
&2\int d^2\alpha e^{z\alpha^*-z^*\alpha}\Delta(\alpha,\alpha^*)\\
&=\frac{1}{\pi}2\int d^2\alpha e^{z\alpha^*-z^*\alpha}:e^{-2(a^\dagger-\alpha^*)(a-\alpha)}:\\
&=e^{za^\dagger}e^{-z^*a-|z|^2/2}\equiv D(z),
\end{aligned} \tag{21}$$

$D(z)$ is a displacement operator, conversely,

$$\int\frac{d^2z}{2\pi^2}e^{z^*\alpha-z\alpha^*}D(z)=\Delta(\alpha,\alpha^*). \tag{22}$$

Using Eqs. (12) and (10) we now calculate

$$\begin{aligned}
&\int_{-\infty}^{\infty}dxe^{ix}|x,\mu,\nu\rangle\langle x,\mu,\nu|\\
&=\iint_{-\infty}^{\infty}dx'dp'\Delta(p',x')e^{i(\mu x'+\nu p')}\\
&=2\int d^2\alpha\Delta(\alpha)e^{\alpha^*\eta-\alpha\eta^*}=D(\eta),
\end{aligned} \tag{23}$$

where we have used $i(\mu x'+\nu p') = \alpha^*\eta-\alpha\eta^*$,

$$\begin{aligned}
\eta&=\frac{-\nu+i\mu}{\sqrt{2}},\quad \eta^*=\frac{-\nu-i\mu}{\sqrt{2}},\\
\mu&=\frac{\eta-\eta^*}{\sqrt{2}i},\quad \nu=-\frac{\eta+\eta^*}{\sqrt{2}}.
\end{aligned} \tag{24}$$

Performing trace for both sides of Eq. (23) multiplied by ρ yields

$$\begin{aligned}
&\mathrm{Tr}\left[\rho\int_{-\infty}^{\infty}dxe^{ix}|x,\mu,\nu\rangle\langle x,\mu,\nu|\right]\\
&=\mathrm{Tr}[\rho D(\eta)]\equiv\chi(\eta)|_{\eta=\frac{-\nu+i\mu}{\sqrt{2}}},
\end{aligned} \tag{25}$$

where χ is just the characteristic function of the density operator often used in tackling quantum master equations,[11] or using Eq. (12) we see Eq. (25) as

$$\int_{-\infty}^{\infty}dxe^{ix}\mathfrak{T}(x,\mu,\nu)=\chi(\eta)|_{\eta=\frac{-\nu+i\mu}{\sqrt{2}}}, \tag{26}$$

this is the new relation between the tomogram and the characteristic function whose parameter is composed of (μ,ν), the tomograph parameters. This relation seems first observed in the present work.

Recall that the Weyl classical correspondence of ρ is $2\pi\mathrm{Tr}[\rho\Delta(\alpha,\alpha^*)]$, the Weyl quantization rule can be expressed as

$$\rho=4\pi\int d^2\alpha\Delta(\alpha,\alpha^*)\mathrm{Tr}[\rho\Delta(\alpha,\alpha^*)]. \tag{27}$$

Substituting Eq. (22) into Eq. (27) and using Eq. (21) yield

$$\begin{aligned}
\rho&=2\int d^2\alpha\int\frac{d^2z}{\pi}e^{\eta^*\alpha-\eta\alpha^*}D(\eta)\mathrm{Tr}[\rho\Delta(\alpha,\alpha^*)]\\
&=\int\frac{d^2\eta}{\pi}D(\eta)\mathrm{Tr}\left[2\rho\int d^2\alpha e^{\eta^*\alpha-\eta\alpha^*}\Delta(\alpha,\alpha^*)\right]\\
&=\int\frac{d^2\eta}{\pi}D(\eta)\mathrm{Tr}[\rho D(-\eta)]\\
&=\int\frac{d^2\eta}{\pi}D(-\eta)\chi(\eta).
\end{aligned} \tag{28}$$

From Eqs. (28) and (26) we have

$$\begin{aligned}
\rho&=\frac{1}{\pi}\int d^2\eta e^{a\eta^*-a^\dagger\eta}\chi(\eta)\\
&=\frac{1}{2\pi}\int d\nu d\mu e^{-i(\mu X+\nu P)}\chi(\eta)|_{\eta=\frac{-\nu+i\mu}{\sqrt{2}}}\\
&=\frac{1}{2\pi}\iiint d\nu d\mu dx\mathfrak{T}(x,\mu,\nu)e^{i(x-\mu X-\nu P)}.
\end{aligned} \tag{29}$$

This is the inverse relation of $\mathfrak{T}(x,\mu,\nu) \equiv \mathrm{Tr}[\rho|x,\mu,\nu\rangle\langle x,\mu,\nu|]$. We can also derive its invertible relation by using the Weyl ordering form of the Wigner operator[12–14]

$$\Delta(p',x')=\vdots\delta(x'-X)\delta(p'-P)\vdots, \tag{30}$$

i.e.,

$$\begin{aligned}
\rho&=2\pi\iint dx'dp'W(p',x')\Delta(p',x')\\
&=2\pi\iint dx'dp'W(p',x')\vdots\delta(x'-X)\delta(p'-P)\vdots\\
&=\frac{1}{2\pi}\iint d\nu d\mu dx'dp'W(p',x')\vdots e^{i\mu(x'-X)+i\nu(p'-P)}\vdots\\
&=\frac{1}{2\pi}\iint d\nu d\mu dx\iint dx'dp'W(p',x')\\
&\quad\cdot\delta(x-\mu x'-\nu p')e^{i(x-\mu X-\nu P)}\\
&=\frac{1}{2\pi}\int\iint d\nu d\mu dx\mathfrak{T}(x,\mu,\nu)e^{i(x-\mu X-\nu P)}.
\end{aligned} \tag{31}$$

It then follows the expansion of density operator in terms of the intermediate coordinate-momentum representation

$$\begin{aligned}
\rho=\frac{1}{2\pi}\int d\nu d\mu dx|x,\mu,\nu\rangle\\
\cdot\langle x,\mu,\nu|e^{-ix}\chi(\eta)|_{\eta=\frac{-\nu+i\mu}{\sqrt{2}}}.
\end{aligned} \tag{32}$$

In summary, we have revealed a new relation between the tomogram and the characteristic function of the density operator and obtained the new expansion of the density operator in terms of the intermediate coordinate-momentum representation, the tomography theory is thus enriched. The new results

of this work may be applied to the design of a two-dimensional x-ray transmission grating.[15] Throughout our derivations we make full use of the operator formalism and the IWOP technique,[16–18] so that our representation is neat and concise,

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