CHIN. PHYS. LETT. Vol. 26, No. 10 (2009) 100303

Inseparability of Phase-Damped Bipartite Gaussian States *

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(Received 18 December 2008)

For bipartite non-Gaussian states which are prepared by applying the phase damping to Gaussian states, we use Fock subspace inseparable criterion and the Shchukin–Vogel inseparable criterion to analyze their inseparability. The lowest order of the two criteria is obtained. Fock subspace criterion is more efficient in detecting the entanglement of damped Gaussian state by numerical results. Two photon Fock subspace criterion does not improve the entanglement detecting condition.

PACS: 03. 65. Ud, 03. 67. Mn, 42. 50. Dv

The entanglement or inseparability of Gaussian states has been widely studied.[1–5] Recently non-Gaussian states and non-Gaussian operations have attracted much attention.[6–8] The criterion based on the moments (thus commutation relations) on the inseparability of continuous variable states was recently given by Shchukin and Vogel.[9] Another series of negative partial transposition (NPT) conditions were obtained in Fock space representation.[10] The NPT inseparability of any phase damped two-mode $x-p$ symmetric Gaussian states has been completely solved.[11] The form of single photon Fock subspace inseparable criterion has been presented[10] for general phase damped bipartite Gaussian states. However, it is not known whether the entanglement detecting performance of Fock criterion is better than the Shchukin–Vogel criterion for general phase damped bipartite Gaussian states. It is also not clear whether the single photon Fock subspace inseparable criterion can be improved further by using two photon Fock basis for general phase damped Gaussian state, although it is known that the single photon Fock subspace criterion cannot be improved for any phase damped two-mode $x-p$ symmetric Gaussian states. These problems will be addressed in this Letter.

A 1×1 bipartite Gaussian state ρ_G can be characterized with its correlation matrix (CM) which can be transform to the standard form

$$\alpha=\begin{bmatrix} m & k_x \\ k_x & n \end{bmatrix}\oplus\begin{bmatrix} m & -k_p \\ -k_p & n \end{bmatrix}, \tag{1}$$

by local operations,[5] with $k_x k_p > 0$ for the state being necessarily inseparable.[3] Further, $k_x > 0$ is assumed for convenience.

The characteristic function of a state ρ is $\chi(\mu)=\mathrm{tr}[\rho\mathcal{D}(\mu)]$, where $\mathcal{D}(\mu)=\exp(\mu a'^{\dagger}-\mu^* a')$ is the displacement operator, with $\mu=[\mu_a,\mu_b]$ being a complex vector, $a'=[a,b]^T$ is the vector of annihilation operators. Inversely, $\rho=\int\frac{d^2\mu_a d^2\mu_b}{\pi^2}\chi(\mu)\mathcal{D}(-\mu)$. The characteristic function of a Gaussian state ρ_G with the standard form CM α is

$$\chi_G(\mu)=\exp\big[-m|\mu_a|^2-n|\mu_b|^2+c_1(\mu_a\mu_b+\mu_a^*\mu_b^*)+c_2(\mu_a\mu_b^*+\mu_a^*\mu_b)\big], \tag{2}$$

where $c_1=\frac{1}{2}(k_p+k_x)$, $c_2=\frac{1}{2}(k_p-k_x)$.

In the presence of phase-damping, the density matrix obeys the master equation[12,13]

$$\frac{d\rho}{dt}=\frac{\gamma_a}{2}L[a^\dagger a]\rho+\frac{\gamma_b}{2}L[b^\dagger b]\rho, \tag{3}$$

where the Lindblad super-operators are defined as $L[\hat{o}]\rho\equiv 2\hat{o}\rho\hat{o}^\dagger-\hat{o}^\dagger\hat{o}\rho-\rho\hat{o}^\dagger\hat{o}$, and γ_a,γ_b are the phase damping coefficients of the two modes. In the evolution of the state, we do not consider the actions of amplitude damping, amplification and thermal noise to the initial Gaussian state, for the Gaussian character of the state is preserved with these factors. Let the initial state be the Gaussian state

$$\rho_G=\sum_{i_a,i_b,j_a,j_b}C_{i_a i_b,j_a j_b}(0)|i_a i_b\rangle\langle j_a j_b|, \tag{4}$$

then it can be verified that the phase-damped state will be

$$\rho=\sum_{i_a,i_b,j_a,j_b}C_{i_a i_b,j_a j_b}(t)|i_a i_b\rangle\langle j_a j_b|, \tag{5}$$

with

$$C_{i_a i_b,j_a j_b}(t)=C_{i_a i_b,j_a j_b}(0)\exp\Big(-\sum_{l=a,b}\frac{\gamma_l}{2}|i_l-j_l|^2 t\Big). \tag{6}$$

In the verification of Eq. (3) we have used the fact that $L[a^\dagger a](|i\rangle\langle j|)=-(i-j)^2|i\rangle\langle j|$.

Let us consider the one-photon subspace spanned by Fock space vectors $|01\rangle$, $|10\rangle$, then one of the sufficient condition of inseparability for any bipartite state ρ can be[10]

$$\langle 01|\rho^{PT}|01\rangle\langle 10|\rho^{PT}|10\rangle<\langle 01|\rho^{PT}|10\rangle\langle 10|\rho^{PT}|01\rangle, \tag{7}$$

*Supported by the National Natural Science Foundation of China under Grant No 10575092.

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where ρ^{PT} is the partial transpose of ρ. It follows that

$$C_{01,01}(0)C_{10,10}(0) < C_{00,11}(0)C_{11,00}(0)e^{-2\bar{\gamma}t}, \quad (8)$$

where $\bar{\gamma} = \frac{1}{2}(\gamma_a + \gamma_b)$. The coefficients $C_{i_a i_b, j_a j_b}(0)$ of the bipartite Gaussian state ρ_G can be evaluated with its characteristic function, we have

$$C_{i_a i_b, j_a j_b}(0) = \int \frac{d^2\mu_a d^2\mu_b}{\pi^2} \chi_G(\mu) \langle i_a i_b | \mathcal{D}(-\mu) | j_a j_b \rangle. \quad (9)$$

Note that

$$\langle 01|\mathcal{D}(-\mu)|01\rangle = (1-|\mu_b|^2)\exp\Big[-\frac{1}{2}|\mu|^2\Big], \quad (10)$$

$$\langle 10|\mathcal{D}(-\mu)|10\rangle = (1-|\mu_a|^2)\exp\Big[-\frac{1}{2}|\mu|^2\Big], \quad (11)$$

$$\langle 00|\mathcal{D}(-\mu)|11\rangle = \mu_a^*\mu_b^*\exp\Big[-\frac{1}{2}|\mu|^2\Big]. \quad (12)$$

Let

$$\begin{aligned} I &= \int \frac{d^2\mu_a d^2\mu_b}{\pi^2} \chi_G(\mu) e^{-\frac{1}{2}|\mu|^2} \\ &= \int \frac{d^2\mu_a d^2\mu_b}{\pi^2} \exp[-m'|\mu_a|^2 - n'|\mu_b|^2 \\ &\quad + c_1(\mu_a\mu_b + \mu_a^*\mu_b^*) + c_2(\mu_a\mu_b^* + \mu_a^*\mu_b)], \end{aligned} \quad (13)$$

with $m' = m + \frac{1}{2}$, $n' = n + \frac{1}{2}$. By using the integral formula

$$\begin{aligned} &\int \frac{d^2\nu}{\pi} \exp\left[-\frac{1}{2}(\nu,\nu^*)\begin{pmatrix} A & B \\ C & D \end{pmatrix}(\nu,\nu^*)^T\right] \\ &= \left(\det\begin{bmatrix} C & D \\ A & B \end{bmatrix}\right)^{-\frac{1}{2}}, \end{aligned} \quad (14)$$

we have $I = \frac{1}{\sqrt{K}}$, where $K = (m'n' - k_x^2)(m'n' - k_p^2)$. Then

$$C_{01,01}(0) = I + \frac{\partial I}{\partial n'} = \frac{1}{\sqrt{K}}(1 - B_{11}), \quad (15)$$

$$C_{10,10}(0) = I + \frac{\partial I}{\partial m'} = \frac{1}{\sqrt{K}}(1 - B_{00}), \quad (16)$$

with

$$B_{11} = \frac{m'}{K}\Big[m'n' - \frac{1}{2}(k_x^2 + k_p^2)\Big], \quad (17)$$

$$B_{00} = \frac{n'}{K}\Big[m'n' - \frac{1}{2}(k_x^2 + k_p^2)\Big]. \quad (18)$$

To evaluate $C_{00,11}(0)$, we calculate the integral

$$\begin{aligned} W &= \int \frac{d^2\mu_a d^2\mu_b}{\pi^2} \exp\big[-m'|\mu_a|^2 - n'|\mu_b|^2 \\ &\quad + c^*\mu_a\mu_b + c\mu_a^*\mu_b^* + c_2(\mu_a\mu_b^* + \mu_a^*\mu_b)\big]. \end{aligned} \quad (19)$$

With formula (14), the integral is $W = [(m'n' - |c|^2 - c_2^2)^2 - 4|c|^2c_2^2]^{-\frac{1}{2}}$.

Then

$$C_{00,11}(0) = \frac{\partial W}{\partial c}\Big|_{c=c^*=c_1} = \frac{B_{01}}{\sqrt{K}}, \quad (20)$$

with

$$B_{01} = \frac{k_x + k_p}{2K}(m'n' - k_x k_p). \quad (21)$$

The single photon Fock subspace inseparability criterion for the phase-damped state is

$$(1 - B_{00})(1 - B_{11}) < e^{-2\bar{\gamma}t}B_{01}^2. \quad (22)$$

Also, it is not difficult to obtain the submatrix of ρ^{PT} in the two photon basis $|02\rangle$, $|11\rangle$, $|20\rangle$. The inseparable condition is the negativity of the following matrix

$$M_2 = \begin{bmatrix} f_{1b} & f_{2b} & f_3 \\ f_{2b} & f_4 & f_{2a} \\ f_3 & f_{2a} & f_{1a} \end{bmatrix}, \quad (23)$$

where $f_{1a} = I + 2\frac{\partial I}{\partial m'} + \frac{1}{2}\frac{\partial^2 I}{\partial m'^2} = \frac{1}{\sqrt{K}}\big(1 - B_{00} + \frac{3}{2}B_{00}^2 - \frac{n'^2}{2K}\big)$, $f_{2a} = e^{-\bar{\gamma}t}\big(\sqrt{2}\frac{\partial W}{\partial c} + \frac{1}{\sqrt{2}}\frac{\partial^2 W}{\partial c \partial m'}\big)\big|_{c=c^*=c_1} = \frac{e^{-\bar{\gamma}t}}{\sqrt{2K}}\big(2B_{01} - 3B_{01}B_{00} + \frac{(k_x+k_p)n'}{2K}\big)$, with the substitution of $m' \leftrightarrow n'$, we obtain f_{1b}, f_{2b}, and $f_3 = \frac{1}{2}e^{-4\bar{\gamma}t}\frac{\partial^2 W}{\partial c^2}\big|_{c=c^*=c_1} = \frac{e^{-4\bar{\gamma}t}}{2\sqrt{K}}\big[3B_{01}^2 - \frac{(k_x+k_p)^2}{4K}\big]$, $f_4 = I + \frac{\partial I}{\partial m'} + \frac{\partial I}{\partial n'} + \frac{1}{2}\frac{\partial^2 I}{\partial m' \partial n'} = \frac{1}{\sqrt{K}}\big(1 - B_{00} - B_{11} + 3B_{00}B_{11} - \frac{4m'n' - k_x^2 - k_p^2}{2K}\big)$. We calculate $\det(M_2)$ at time $t_2 = t_{F1}/2$, with

$$t_{F1} = \frac{1}{2\bar{\gamma}}\log\frac{B_{01}^2}{(1 - B_{00})(1 - B_{11})}, \quad (24)$$

being the critical time of the one photon Fock subspace criterion. The positivity of $\det(M_2)$ in Fig. 1 shows that the entanglement of the damped state can not be detected by two photon Fock subspace inseparable criterion even at $t_2 = t_{F1}/2$, there is no improvement of the inseparable criterion by using two-photon subspace criterion.

For the special case of $k_p = k_x = k$,[11] the initial Gaussian state is $x - p$ symmetric,[14] the two photon Fock subspace criterion can be analytically solved. Denote $F = m'n' - k^2$, we have $f_{1a} = \frac{1}{F}\big(1 - \frac{n'}{F}\big)^2$, $f_{1b} = \frac{1}{F}\big(1 - \frac{m'}{F}\big)^2$, $f_{2a} = \frac{\sqrt{2}k}{F^2}\big(1 - \frac{n'}{F}\big)e^{-\bar{\gamma}t}$, $f_{2b} = \frac{\sqrt{2}k}{F^2}\big(1 - \frac{m'}{F}\big)e^{-\bar{\gamma}t}$, $f_3 = \frac{k^2}{F^3}e^{-4\bar{\gamma}t}$, $f_4 = \frac{1}{F}\big[(1 - \frac{n'}{F})(1 - \frac{m'}{F}) + \frac{k^2}{F^2}\big]$. The determinant can be simplified to

$$\det(M_2) = 2A^3\begin{vmatrix} 1 & xd & x^4d^2 \\ xd & \frac{1}{2}(1+d^2) & xd \\ x^4d^2 & xd & 1 \end{vmatrix}, \quad (25)$$

with $A = \frac{1}{F}(1 - \frac{m'}{F})(1 - \frac{n'}{F})$, $x = e^{-\bar{\gamma}t}$, $d = \frac{k}{\sqrt{(F-m')(F-n')}}$. We have $\det(M_2) = A^3[(1+d^2)(1 - x^8d^4) + 2(x^4d^2 - 1)x^2d^2)]$.

The only reasonable solution to $\det(M_2(x)) = 0$ is

$$x_F = 1/\sqrt{d}. \quad (26)$$

In the case of $k_p = k_x = k$, we have $t_{F1} = \frac{1}{2\bar{\gamma}}\log\frac{k^2}{(F-m')(F-n')}$, thus $d = e^{\bar{\gamma}t_{F1}}$, and $x_F \equiv$

$e^{-\bar{\gamma}t_{F2}} = e^{-\frac{1}{2}\bar{\gamma}t_{F1}}$. So $t_{F2} = \frac{1}{2}t_{F1}$, the critical time for two photon Fock subspace criterion is only half the critical time of one photon Fock subspace criterion for two-mode $x-p$ symmetric Gaussian phase-damped state.

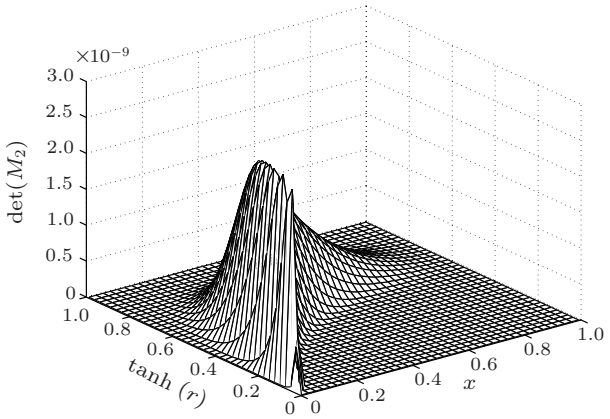

Fig. 1. Determinant of partial transposed density matrix in two-photon Fock subspace with $t_2 = t_{F1}/2$, $\gamma_A = 1.5$, $\gamma_B = 1.6$.

Since the first moments of the initial Gaussian state are set to be 0, we need only consider the second moments of the phase-damped state. For $\chi(\mu) = \mathrm{Tr}[\rho D(\mu)] = \mathrm{Tr}[\rho e^{-\frac{1}{2}|\mu|^2} e^{\mu_a a^\dagger + \mu_b b^\dagger} e^{-\mu_a^* a - \mu_b^* b}]$, we have

$$\chi'(\mu) = \chi(\mu) e^{\frac{1}{2}|\mu|^2} = \mathrm{Tr}[\rho e^{\mu_a a^\dagger + \mu_b b^\dagger} e^{-\mu_a^* a - \mu_b^* b}]. \quad (27)$$

The expression of $\chi(\mu)$ for phase damped Gaussian state is[15]

$$\chi(\mu) = \frac{1}{2\pi t\sqrt{\gamma_a \gamma_b}} \int_{-\infty}^{+\infty} \exp\Big[-\frac{u^2}{2\gamma_a t} - \frac{v^2}{2\gamma_b t}\Big] \cdot \chi_G(\mu_a e^{iu}, \mu_b e^{iv}) du dv. \quad (28)$$

By differentiating $\chi'(\mu)$ with respect to the parameters μ_i or μ_i^*, then setting all μ_i and μ_i^* to be 0, one has

$$\langle a^\dagger a\rangle = \mathrm{Tr}(\rho a^\dagger a) = -\frac{\partial^2 \chi'(\mu)}{\partial \mu_a \partial \mu_a^*}\Big|_{\mu=\mu^*=0} = m - \frac{1}{2}, \quad (29)$$

$$\langle a^\dagger b^\dagger\rangle = \frac{\partial^2 \chi'(\mu)}{\partial \mu_a \partial \mu_b}\Big|_{\mu=\mu^*=0} = -c_1 e^{-\bar{\gamma}t}, \quad (30)$$

$$\langle a^\dagger b\rangle = -\frac{\partial^2 \chi'(\mu)}{\partial \mu_a \partial \mu_b^*}\Big|_{\mu=\mu^*=0} = c_2 e^{-\bar{\gamma}t}, \quad (31)$$

and $\langle a^{\dagger 2}\rangle = \langle a^2\rangle = \langle b^{\dagger 2}\rangle = \langle b^2\rangle = 0$, $\langle b^\dagger b\rangle = n - 1/2$, $\langle ab\rangle = -c_1 e^{-\bar{\gamma}t}$, $\langle ab^\dagger\rangle = c_2 e^{-\bar{\gamma}t}$. The inseparability criterion based on the second moments is $D_5 < 0$, with

$$D_5 = \det \begin{bmatrix} m-\frac{1}{2} & 0 & -c_1 e^{-\bar{\gamma}t} & c_2 e^{-\bar{\gamma}t} \\ 0 & m+\frac{1}{2} & c_2 e^{-\bar{\gamma}t} & -c_1 e^{-\bar{\gamma}t} \\ -c_1 e^{-\bar{\gamma}t} & c_2 e^{-\bar{\gamma}t} & n-\frac{1}{2} & 0 \\ c_2 e^{-\bar{\gamma}t} & -c_1 e^{-\bar{\gamma}t} & 0 & n+\frac{1}{2} \end{bmatrix}. \quad (32)$$

Denote $Y = e^{-2\bar{\gamma}t}$, the inequality $D_5 < 0$ can be written as

$$k_x^2 k_p^2 Y^2 - [k_x k_p/2 + (k_x^2 + k_p^2)mn]Y + \Big(m^2 - \frac{1}{4}\Big)\Big(n^2 - \frac{1}{4}\Big) < 0. \quad (33)$$

The solution should be $Y_0 < Y < Y_1$, where Y_0 and Y_1 are the roots of the corresponding quadrature equation. Based on the second moments, one has the Shchukin–Vogel inseparability condition

$$\bar{\gamma}t < -\frac{1}{2}\log Y_0. \quad (34)$$

To compare the performance of the two criteria applying on the phase-damped Gaussian state, we need firstly generate a two-mode Gaussian state as the initial state. For simplicity, we assume that the initial Gaussian state is already in its standard form with CM of Eq. (1). The CM α should fulfill a matrix inequality (generalized Robertson uncertainty relation), otherwise the corresponding state may not be a physical state. Thus the positive parameters m, n, k_x, k_p cannot be randomly chosen. It has been proved[16] that any standard form Gaussian state can also be represented with at most four free chosen parameters. The Gaussian states are classified into four kinds.[16] We here use the first kind of Gaussian states as the initial states to compare the two inseparability criteria (other kinds of Gaussian initial states can similarly be treated). The standard form the Gaussian state is generated as follows: (1) The first two free parameters are the symplectic eigenvalues γ_A and γ_B ($\gamma_A, \gamma_B \geq \frac{1}{2}$). (2) The third free parameter is $x > 0$, which is used as the parameter of local squeezing before the two-mode squeezing. (3) The last parameter is the two mode squeezing parameter r. (4) The state generated has a CM $\alpha_0 = \begin{bmatrix} m_1 & k_1 \\ k_1 & n_1 \end{bmatrix} \oplus \begin{bmatrix} m_2 & -k_2 \\ -k_2 & n_2 \end{bmatrix}$, where $m_1 = \gamma_A x^{-1}\cosh^2 r + \gamma_B x \sinh^2 r$, $m_2 = \gamma_A x \cosh^2 r + \gamma_B x^{-1}\sinh^2 r$, $n_1 = \gamma_A x^{-1}\sinh^2 r + \gamma_B x \cosh^2 r$, $n_2 = \gamma_A x \sinh^2 r + \gamma_B x^{-1}\cosh^2 r$, $k_1 = \frac{1}{2}(\gamma_A x^{-1} + \gamma_B x)\sinh 2r$, $k_2 = \frac{1}{2}(\gamma_A x + \gamma_B x^{-1})\sinh 2r$. This CM is not in the standard form. (5) The standard form can be obtained by local operations, it is in the form of (1) with $m = \sqrt{m_1 m_2}$, $n = \sqrt{n_1 n_2}$, $k_x = k_1\big(\frac{m_2 n_2}{m_1 n_1}\big)^{\frac{1}{4}}$, $k_p = k_2\big(\frac{m_1 n_1}{m_2 n_2}\big)^{\frac{1}{4}}$.

We calculate the critical time of the state according to the lowest order of Fock and Shchukin-Vogel criteria. The critical time of a criterion is such a time in the damping process that NPT inequality reaches its equality. If the evolution time is less than the critical time (t_F and t_{SV} for the two criteria, respectively), the phase-damped state is detected to be entangled. The longer the critical time, the stronger the ability of detecting the entanglement. The ratio $(\frac{t_{SV}}{t_F})$ of the critical time of the two criteria are shown in Fig. 2. The critical time of the Shchukin-Vogel criterion is

less than the critical time of the Fock subspace criterion. The largest time ratio is less than 0.2 as shown in Fig. 2. We have calculated the time ratio many times with other symplectic eigenvalues γ_A and γ_B. The ratios are always less than 1. Hence, at the critical time t_{SV} of the Shchukin-Vogel criterion, the entanglement can not be detected with the Shchukin-Vogel criterion, but it can still be detected with the Fock subspace criterion for $t_F > t_{SV}$ in most situations. We can see that the Fock subspace criterion is more efficient in detecting the entanglement.

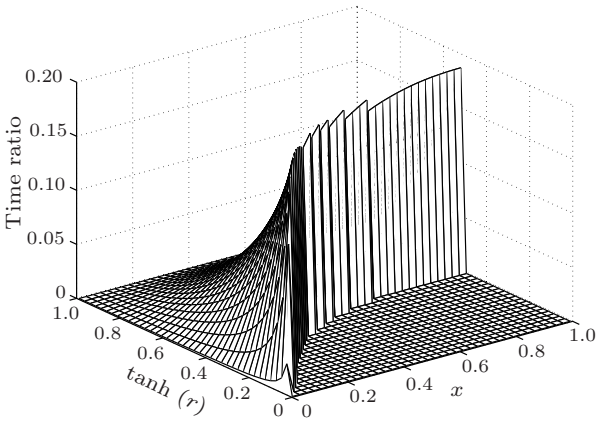

Fig. 2. Time ratio is ratio of the Shchukin-Vogel critical time with respect to the Fock critical time. The separable region for the initial state is specified by the zero region. $\gamma_A = 1.5$, $\gamma_B = 1.6$.

In summary, we have derived the single and two photon Fock subspace inseparable conditions and second moment Shchukin-Vogel inseparable condition for the phase-damped standard form bipartite Gaussian states. The extension of the results for any bipartite Gaussian state is straightforward. The numeric comparison of the conditions shows that the one photon Fock subspace condition is more efficient in detecting the entanglement of the phase-damped Gaussian state.

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