CHIN. PHYS. LETT. Vol. 26, No. 1 (2009) 010301

Entanglement Dynamics of a Three-Level Atom in Ξ Configuration *

XU Hang-Shi(徐航时), XU Jing-Bo(许晶波)**

Zhejiang Institute of Modern Physics and Physics Department, Zhejiang University, Hangzhou 310027

(Received 19 September 2008)

Dynamical behaviour of entanglement between the cavity mode and a three-level atom in the Ξ configuration is studied by making use of log-negativity. The influence of initial temperature of the thermal field and detuning on the entanglement is investigated. It is found that the maximal values of entanglement decreases with the temperature of the thermal field and can be controlled by the detuning.

PACS: 03.65.Ud, 03.67.−a, 42.50.Dv

Quantum entanglement is of great interest in the various fields of quantum information, such as quantum computation,[1] quantum cryptography,[2] quantum teleportation,[3,4] entanglement swapping,[5,6] and quantum communication.[7] In recent years, the manipulation of quantum entanglement for the system of the cavity quantum electrodynamics (QED) has been extensively investigated.[8−13] It has been shown that the entanglement can be generated in the interaction of a two-level atom with a cavity field in a thermal state.[8−11]

Recently, much attention has been focused on the study of N-level quantum systems to implement quantum information processing. For example, key distribute protocols based on entangled three-level quantum systems are proposed.[14] It was pointed out that quantum key distribution schemes using three-level systems are more secure against symmetric attacks than protocols using two-level systems.[15] Three-level systems are also used to implement quantum phase gates and the controlled-NOT gates.[16,17]

Recently, the entanglement in cavity QED consisting of a three-level atom and vacuum fields has been discussed.[18,19] In this Letter, we investigate the dynamical behaviour of entanglement between the cavity mode and three-level atom in the Ξ configuration by adopted log-negativity as a measure. The influence of initial temperature of the thermal field and various systemic parameters on the entanglement is also discussed.

Now, we consider a three-level atom in the Ξ configuration interacting with a single mode field. The Hamiltonian for this model can be described ($\hbar = 1$) by[20]

$$H = \sum_{1=1}^{3} \varepsilon_i \sigma_{ii} + \omega a^{\dagger} a + g_1 a^{l_1} \sigma_{31} + g_1 a^{\dagger l_1} \sigma_{13} + g_2 a^{l_2} \sigma_{23} + g_2 a^{\dagger l_2} \sigma_{32}, \tag{1}$$

where the levels $|1\rangle$ is ground-state and the levels $|2\rangle$ is the up excited-state and $|3\rangle$ is the mediate state. $\sigma_{21} = |2\rangle\langle 1|$ and $\sigma_{12} = |1\rangle\langle 2|$ are atomic transition operators and $\sigma_{ii} = |1\rangle\langle 1|(i = 1, 2, 3)$ are three population operators of the atom. Here the interactions are generalized to be multi-photon (l_1, l_2) forms. We introduce the detuning $\Delta_i = \varepsilon_3 - \varepsilon_i + (-1)^i l_i \omega$,(i=1,2) and consider the case of $\Delta_1 = \Delta_2 = \Delta$ for simplicity.

It is easy to verify that there exist one constant of motion in the Hamiltonian (1),

$$K = a^{\dagger} a + l_1(1 - \sigma_{11}) + l_2 \sigma_{22}, \tag{2}$$

which commutes with Hamiltonian (1). We then introduce the generators of the SU(3) algebra

$$\begin{aligned} &S_{ii} = \sigma_{ii} (i = 1.2.3);\ S_{12} = \frac{a^{\dagger l_1} a^{\dagger l_2} \sigma_{12}}{\sqrt{F_1(K) F_2(K - l_1)}};\\ &S_{13} = \frac{a^{\dagger l_1} \sigma_{13}}{\sqrt{F_1(K)}};\quad S_{23} = \frac{a^{l_2} \sigma_{23}}{\sqrt{F_2(K - l_1)}};\\ &S_{21} = S_{12}^{\dagger};\ S_{32} = S_{23}^{\dagger};\ S_{31} = S_{13}^{\dagger}, \end{aligned} \tag{3}$$

where

$$F_1(K) = \frac{K!}{(K - l_1)!};\quad F_2(K) = \frac{K!}{(K - l_2)!}, \tag{4}$$

and $S_{ij} (i, j = 1, 2, 3)$ are the generators of the $SU(3)$ algebra, they satisfy the following commutation and anticommutation relations:[21]

$$\begin{aligned} &[S_{ij}, S_{kl}] = S_{il}\delta_{jk} - S_{kj}\delta_{il},\\ &\{S_{ij}, S_{kl}\} = S_{il}\delta_{jk} + S_{kj}\delta_{il},\quad \sum_{i=1}^{3} S_{ii} = 1. \end{aligned} \tag{5}$$

In terms of the SU(3) generators, we can rewrite the Hamiltonian (1) as

$$\begin{aligned} H =& \varepsilon + \omega K + \Delta S_{33} + g_1 \sqrt{F_1(K)}(S_{31} + S_{13})\\ &+ g_2 \sqrt{F_2(K - l_1)}(S_{23} + S_{32}), \end{aligned} \tag{6}$$

where $\varepsilon = \frac{1}{2}\varepsilon_1 + \varepsilon_2 - l_1\omega - l_2\omega$. By making use of the dynamical algebraic approach,[21] we can find the time evolution operator[21]

$$U(t) = \frac{S_{33}}{2}[1 + \cos\theta_2(K)] \exp(-i f_2 t) + [1 - \cos\theta_2(K)]$$

*Supported by the National Natural Science Foundation of China under Grant No 10774143, and the National Basic Research Programme of China under Grant No 2006CB921403.

**Email: xujb@zju.edu.cn

© 2009 Chinese Physical Society and IOP Publishing Ltd

$$\cdot \exp(-if_3t)+S_{22}\Big[\cos^2\theta_1(K)\exp(-if_1t)+\frac{1}{2}(1 - \cos\theta_2(K))\sin^2\theta_1(K)\exp(-if_2t)+\frac{1}{2}(1+\cos\theta_2(K)) \cdot \sin^2\theta_1(K)\exp(-if_3t)\Big]+S_{11}\Big[\sin^2\theta_1(K)\exp(-if_1t) + \frac{1}{2}(1-\cos\theta_2(K))\cos^2\theta_1(K)\exp(-if_2t)+\frac{1}{2}(1 + \cos\theta_2(K))\cos^2\theta_1(K)\exp(-if_3t)\Big]+\frac{1}{2}(S_{12}+S_{21}) \cdot \sin 2\theta_1(K)\Big[-\exp(-if_1t)+\frac{1-\cos\theta_2(K)}{2}\exp(-if_2t) + \frac{1+\cos\theta_2(K)}{2}\exp(-if_3t)\Big]+\frac{S_{13}+S_{31}}{2}\sin\theta_2(K) \cdot \cos\theta_1(K)[\exp(-if_2t)-\exp(-if_3t)]+\frac{S_{23}+S_{32}}{2} \cdot \sin\theta_2(K)\sin\theta_1(K)[\exp(-if_2t)-\exp(-if_3t)], \quad (7)$$

where

$$f_1(K)=\varepsilon+\omega K;\quad f_2(K)=\varepsilon+\omega K+\frac{\Delta}{2}+\Omega(K);$$
$$f_3(K)=\varepsilon+\omega K+\frac{\Delta}{2}-\Omega(K);$$
$$\Omega(K)=\sqrt{\Delta/2^2+g_2^2F_2(K-l_1)+g_1^2F_1(K)}, \quad (8)$$

and

$$\theta_1(K)=\tan^{-1}\left[\frac{g_2\sqrt{F_2(K-l_1)}}{g_1\sqrt{F_1(K)}}\right];$$
$$\theta_2(K)=\tan^{-1}\left[\frac{2\sqrt{g_2^2F_2(K-l_1)+g_1^2F_1(K)}}{\Delta}\right]. \quad (9)$$

We now turn to study the dynamical behaviour of the entanglement between the cavity mode and three-level atom in the Ξ configuration. Firstly, we assume that the single mode field is initially prepared in the Fock state $|n\rangle$, and the atom is prepared in the state $|i\rangle (i=1,2,3)$. Then, the density matrix $\rho_i(n,t)$ of time evolution can be written as

$$\rho_i(n,t)=U(t)(|i\rangle|n\rangle)(\langle i|\langle n|)U^\dagger(t) = \sum_{j,k=1}^{3}C_{ij}(n)C_{ik}^*(n)|j\rangle\langle k|\otimes|n+l_{ij}\rangle\langle n+l_{ik}|, \quad (10)$$

where $l_{11}=l_{22}=l_{33}=0$, $l_{21}=-l_{12}=l_1+l_2$, $l_{31}=-l_{13}=l_1$, $l_{23}=-l_{32}=l_2$,

$$C_{11}(m) = \sin^2\theta_1(m)e^{-if_1(m)t}+\frac{1}{2}(1-\cos\theta_2(m)) \cdot \cos_2\theta_1(m)e^{-if_2(m)t}+\frac{1}{2}(1+\cos\theta_2(m)) \cdot \cos_2\theta_1(m)e^{-if_3(m)t}; \quad (11)$$

$$C_{12}(m) = \frac{\sin 2\theta_1(m)}{2}\Big[-e^{-if_1(m)t}+\frac{1-\cos\theta_2(m)}{2} \cdot e^{-if_2(m)t}+\frac{1+\cos\theta_2(m)}{2}e^{-if_3(m)t}\Big]; \quad (12)$$

$$C_{13}(m) = \frac{1}{2}\sin\theta_2(m)\cos\theta_1(m)\big[e^{-if_2(m)t} - e^{-if_3(m)t}\big]; \quad (13)$$

$$C_{21}(m) = C_{12}(m+l_1+l_2); \quad (14)$$

$$C_{22}(m)=\cos^2\theta_1(m+l_1+l_2)e^{-if_1(m+l_1+l_2)t} + \frac{1}{2}\big[1-\cos\theta_2(m+l_1+l_2)\big]\sin^2\theta_1(m+l_1+l_2) \cdot e^{-if_2(m+l_1+l_2)t}+\frac{1}{2}\big[1+\cos\theta_2(m+l_1+l_2)\big] \cdot \sin^2\theta_1(m+l_1+l_2)e^{-if_3(m+l_1+l_2)t}; \quad (15)$$

$$C_{23}(m)=\frac{1}{2}\big[\sin\theta_2(m+l_1+l_2)\sin\theta_1(m+l_1+l_2)\big] \cdot [e^{-if_2(m+l_1+l_2)t}-e^{-if_3(m+l_1+l_2)t}]; \quad (16)$$

$$C_{31}(m)=C_{13}(m+l_1); \quad (17)$$

$$C_{32}(m)=C_{23}(m-l_2); \quad (18)$$

$$C_{33}(m)=\frac{1}{2}[1+\cos\theta_2(m+l_1)]e^{-if_2(m+l_1)t} + \frac{1}{2}[1-\cos\theta_2(m+l_1)]e^{-if_3(m+l_1)t}. \quad (19)$$

In order to quantify the degree of entanglement, we adopt the log-negativity $N(\rho)$ to calculate the entanglement between the three-level atom and the cavity mode. The log-negativity $N(\rho)$ is defined by[22]

$$N(\rho)=\log_2\|\rho^\Gamma\|, \quad (20)$$

where ρ^Γ is the partial transpose of ρ and $\|\rho^\Gamma\|$ denotes the trace norm of ρ^Γ, which is the sum of the singular values of ρ^Γ. It has been proven that the log-negativity is an operational good entanglement measure.[23] If the log-negativity $N(\rho_i(n,t))>0$, the state $\rho_i(n,t)$ must be entangled. It is noted that this is a necessary and sufficient condition for 2×2 and 2×3-dimensional systems, but ceases to be a sufficient condition for higher dimensional systems.[24] The log-negativity $N(\rho_i(n,t))$ characterizing the entanglement between the three-level atom and the cavity mode in the state described by the density matrix $\rho_i(n,t)$ in Eq. (11) can be calculated by

$$N(\rho_i(n,t))=\log_2\|\rho_i^\Gamma(n,t)\|. \quad (21)$$

where

$$\|\rho_i^\Gamma(n,t)\|=1+2\big[|C_{i1}(n)C_{i2}^*(n)|+|C_{i1}(n)C_{i3}^*(n)| + |C_{i2}(n)C_{i3}^*(n)|\big]. \quad (22)$$

In Fig. 1, the log-negativity $N(\rho_2(0,t))$ is plotted as functions of time t and detuning Δ. We take the three-level atom initially in excited-state $|2\rangle$ and the field prepared in the vacuum state. For larger Δ, the period for the fluctuation of the entanglement is shorter and the maximal value of the entanglement is smaller. In order to make it clearer how the detuning Δ influence the entanglement, we show the maximum of the log-negativity $N_{\max}(\rho_2(n))$ along the axis of time t as a function of the detuning Δ in Fig. 2. It is found that the entanglement increase first and then

decrease with the detuning after reach their maxima. $g_1 = g_2 = 1$ have chosen in the numerical calculation.

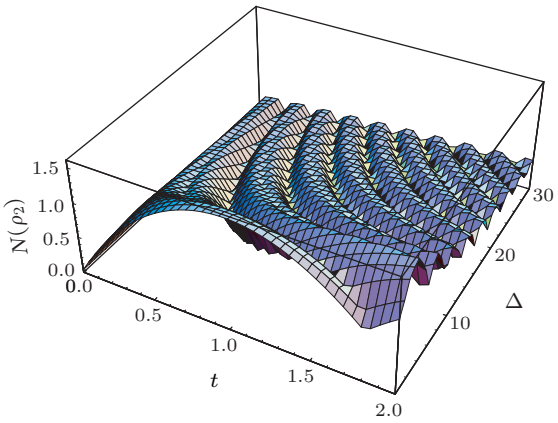


Fig. 1. The log-negativity $N(\rho_2(0,t))$ as a function of time t and detuning Δ with $l_1 = l_2 = 1$.

Next, we assume that the cavity field is initially in an arbitrary thermal state $\sum_{n=0}^{\infty}(1-\alpha)\alpha^n|n\rangle\langle n|, \alpha\epsilon[0,1]$, where $\alpha = \dfrac{\bar{m}}{1+\bar{m}}$, $\bar{m} = \dfrac{1}{e^{\beta\omega}-1}$ and $\bar{m}$ is the mean photon number at the inverse temperature β. The atom is in the mixed-state $\sin^2\theta\cos^2\varphi|1\rangle\langle 1| + \cos^2\theta|2\rangle\langle 2| + \sin^2\theta\sin^2\varphi|3\rangle\langle 3|$.

The initial state of the whole system can be represented by the density matrix $\rho(0)$

$$\rho(0) = \big[\sin^2\theta\cos^2\varphi|1\rangle\langle 1| + \cos^2\theta|2\rangle\langle 2| + \sin^2\theta \cdot \sin^2\varphi|3\rangle\langle 3|\big] \otimes \sum_{n=0}^{\infty}(1-\alpha)\alpha^n|n\rangle\langle n|. \tag{23}$$

In terms of $l_{3k} - l_{3j} = l_{2k} - l_{2j} = l_{1k} - l_{1j}$, the time evolution of $\rho(t)$ can be derived as

$$\rho(t) = U(t)\rho(0)U^+(t) = \sum_{j,k=1}^{3}\sum_{\substack{n=\max(0,\\ l_{1j}-l_{1k})}}^{\infty}\rho_{jk}(n)|j\rangle \cdot \langle k| \otimes |n\rangle\langle n + l_{1k} - l_{1j}|, \tag{24}$$

where

$$\begin{aligned}\rho_{jk}(n) &= (1-\alpha)[\alpha^{n-l_{1j}}\sin^2\theta\cos^2\varphi C_{1j}(n-l_{1j})C^*_{1k} \\ &\cdot (n-l_{1j}) + \alpha^{n-l_{2j}}\cos^2\theta C_{2j}(n-l_{2j}) \\ &\cdot C^*_{2k}(n-l_{2j}) + \alpha^{n-l_{3j}}\sin^2\theta\sin^2\varphi \\ &\cdot C_{3j}(n-l_{3j})C^*_{3k}(n-l_{3j})].\end{aligned} \tag{25}$$

Then, the explicit analytical expression of the log-negativity between the cavity mode and three-level atom can be obtained to be

$$N(\rho) = \log_2\|\rho^\Gamma\| = \log_2\Big[1 + \sum_{m=0}^{l_1-1}(|\delta_m| - \delta_m) + \sum_{m=0}^{\infty}(|\lambda_m| - \lambda_m + |\mu_m| - \mu_m)\Big], \tag{26}$$

where

$$\begin{aligned}\delta_m &= \frac{1}{2}\big\{\rho_{33}(m) + \rho_{22}(m+l_2) - \big\{[\rho_{33}(m) - \rho_{22}(m+l_2)]^2 + 4\rho_{23}(m)\rho_{32}(m+l_2)\big\}^{1/2}\big\}; \\ \lambda_m &= \frac{1}{3}\big[\rho_{11}(m) + \rho_{22}(m+l_1+l_2) + \rho_{33}(m+l_1)\big] + 2\sqrt[3]{r(m)}\cos[\phi(m) + 2\pi/3]; \\ \mu_m &= \frac{1}{3}\big[\rho_{11}(m) + \rho_{22}(m+l_1+l_2) + \rho_{33}(m+l_1)\big] + 2\sqrt[3]{r(m)}\cos[\phi(m) + 4\pi/3], \\ r(m) &= \Big[-\frac{c(m)}{3} + \frac{b^3(m)}{9}\Big]^{3/2}; \\ \phi(m) &= \frac{1}{3}\arccos\big[-q(m)/2r(m)\big], \\ q(m) &= d(m) - \frac{b(m)c(m)}{3} + \frac{2b^3(m)}{27}, \\ b(m) &= -[\rho_{11}(m) + \rho_{22}(m+l_1+l_2) + \rho_{33}(m+l_1)]; \\ c(m) &= \rho_{11}(m)\rho_{22}(m+l_1+l_2) + \rho_{11}(m)\rho_{33}(m+l_1) + \rho_{22}(m+l_1+l_2)\rho_{33}(m+l_1) \\ &\quad - \rho_{32}(m+l_1+l_2)\rho_{23}(m+l_1) - \rho_{13}(m+l_1) \cdot \rho_{31}(m) - \rho_{12}(m+l_1+l_2)\rho_{21}(m); \\ d(m) &= \rho_{11}(m)\rho_{32}(m+l_1+l_2)\rho_{23}(m+l_1) + \rho_{22}(m+l_1+l_2)\rho_{13}(m+l_1)\rho_{31}(m) \\ &\quad + \rho_{33}(m+l_1)\rho_{12}(m+l_1+l_2)\rho_{21}(m) - \rho_{11}(m)\rho_{22}(m+l_1+l_2)\rho_{33}(m+l_1) \\ &\quad - \rho_{13}(m+l_1)\rho_{32}(m+l_1)\rho_{21}(m) - \rho_{12}(m+l_1+l_2)\rho_{31}(m)\rho_{23}(m+l_1).\end{aligned}$$

In Fig. 3, the log-negativity $N(\rho)$ is plotted as a function of t and θ or φ. The larger initial population of ground-state leads to the smaller entanglement. When $\theta = \pi/2$ and $\varphi = 0$, i.e., the initial state of atom is in the ground state $|1\rangle$, the log-negativity reaches zero. When $\theta = 0$, i.e. the initial state of atom is in the up excited-state $|2\rangle$, the maximal value of the entanglement reaches its maximum. As $\theta = \pi/2$ and $\varphi = \pi/2$ are chosen, i.e., the initial state of atom is in the mediate state $|3\rangle$, the maximal value of the entanglement is smaller than as initial state of atom is in the up excited-state. In Fig. 4, we display the log-negativity $N(\rho)$ as a function of time for three different values of temperature of the field. For higher temperature(the larger value of α) of the field,the log-negativity is smaller. However the periods that the entanglement is independent from the temperature. In Fig. 5, the log-negativity $N(\rho)$ is plotted as a function of time for three different values of detuning. For larger detuning, the distant between the neighbouring crests of the entanglement is shorter. And the maximum of the log-negativity is also dependent on the values of detuning. We plot the log-negativity $N(\rho)$ as a function of time t for different multi-photon (l_1, l_2) forms in Fig. 6 and show the influence of the multi-photon process on the maxima and the period for the fluctuation of the entanglement.

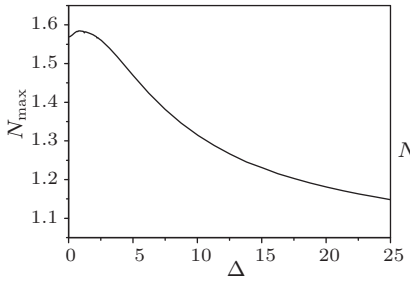


Fig. 2. The maximum of log-negativity $N_{\max}(\rho_2(0))$ as a function of Δ with $l_1 = l_2 = 1$.

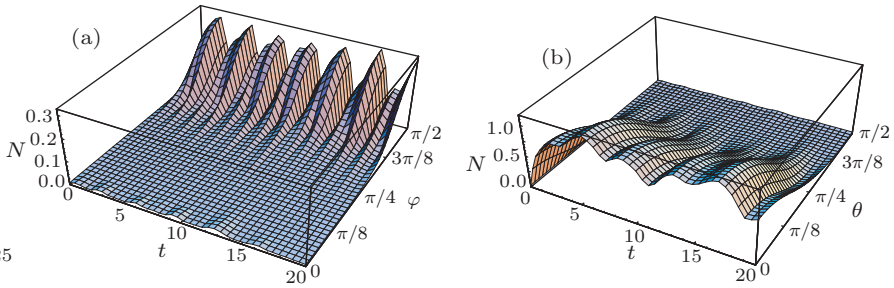


Fig. 3. (a) The log-negativity $N(\rho)$ as functions of t and φ with $\theta = \pi/2$ $l_1 = l_2 = 1$, $\Delta = 10$ and $\alpha = 0.2$. (b) The log-negativity $N(\rho)$ as a function of t and θ with $\varphi = 0$ $l_1 = l_2 = 1$, $\Delta = 10$ and $\alpha = 0.2$.

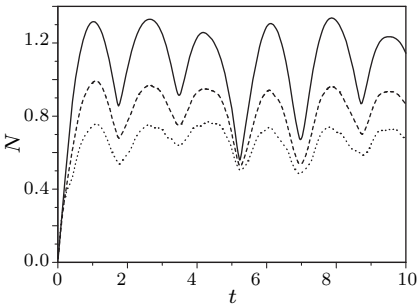


Fig. 4. The log-negativity $N(\rho)$ as a function of time t with l_1=l_2=1, $\theta = 0, \varphi = \pi/2$, $\Delta = 1$ for different temperature of the field, $\alpha = 0.8$(dotted line), $\alpha = 0.5$ (dashed line), $\alpha = 0.2$(solid line).

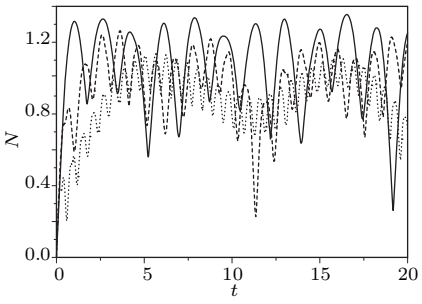


Fig. 5. The log-negativity $N(\rho)$ as a function of time t with l_1=l_2=1, $\theta = 0, \varphi = \pi/2$, $\alpha = 0.2$ for different detuning, $\Delta = 10$(dot line),$\Delta = 5$ (dashed line), $\Delta = 1$ (solid line).

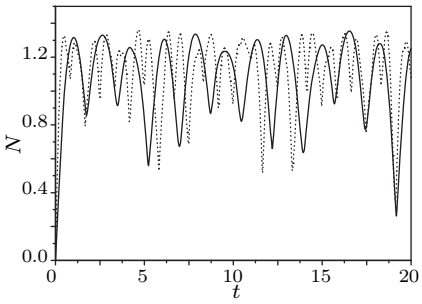


Fig. 6. The log-negativity $N(\rho)$ as a function of time t with $\Delta = 5$, $\theta = 0, \varphi = \pi/2$ and $\alpha = 0.2$ for different multi-photon (l_1, l_2) forms, $l_1 = l_2 = 1$ (solid line), $l_1 = l_2 = 2$ (dotted line).

In summary, we have studied the dynamical behaviours of entanglement between the cavity mode and three-level atom in the Ξ configuration by adopted log-negativity as a measure. It is shown that the maximal values of entanglement decreases with the temperature of the thermal field and can be controlled by the detuning. The period for the fluctuation of the entanglement is sensitive to the detuning and is independent of the temperature of the field. The approach adopted in the present study can be easily extended to the systems of a three-level atom interacting with multi-mode field.

References

[1] Nielsen M A et al 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[2] Kempe J 1999 *Phys. Rev.* A **60** 910
[3] Bennett C H et al 1993 *Phys. Rev. Lett.* **70** 1895
Furusawa A et al 1998 *Science* **282** 706
[4] Braunstein S L et al 1998 *Phys. Rev. Lett.* **80** 869
[5] Polkinghorne R E S et al 1999 *Phys. Rev. Lett.* **83** 2095
[6] Jia X J et al. 2004 *Phys. Rev. Lett.* **93** 259593
[7] Bennett C H et al 1999 *Phys. Rev. Lett.* **83**(15) 3081
Schumacher B et al 1996 *Phys. Rev.* A **54** 2629
[8] Raimond J M et al 2001 *Rev. Mod. Phys.* **73** 565
[9] Bose S et al 2001 *Phys.Rev.Lett.* **87** 05401
[10] Li S B and Xu J B 2003 *Phys. Lett.* A **313** 175
[11] Li W J, Li S B and Xu J B 2004 *Chin. Phys. Lett* **21** 774
[12] Li S B and Xu J B 2005 *Phys. Rev.* A **72** 022332
[13] Bina M et al 2008 *Phys. Rev.* A **77** 033839
[14] Durt T et al 2003 *Phys. Rev.* A **67** 012311
[15] Bruss D et al 2002 *Phys. Rec. Lett.* **88** 127901
[16] Zubairy M S et al 2003 *Phys. Rev.* A **68** 033820
[17] Biswas A and Agarwal G S 2004 *Phys. Rev.* A **69** 062306
[18] Eyob A 2007 *Phys. Rev.* A **76** 023808
[19] Tesfa S 2006 *Phys. Rev.* A **74** 043816
[20] Wu Y and Yang X 1997 *Phys. Rev.* A **56** 2443
[21] Xu J B and Zou X B 1999 *Phys. Rev.* A **60** 4743
[22] Vidal G and Werner R F 2002 *Phys. Rev.* A **65** 032314
[23] Audenaert K et al 2003 *Phys. Rev. Lett.* **90** 027901
[24] Horodecki M et al 1996 *Phys. Lett.* A **223** 1