CHIN.PHYS.LETT. Vol. 23, No. 4 (2006) 832

Dyadic Green Function for an Electromagnetic Medium Inspired by General Relativity

Akhlesh Lakhtakia[1], Tom G. Mackay[2*]

[1]CATMAS, Department of Engineering Science and Mechanics, Pennsylvania State University, University Park, PA 16802–6812, USA

[2]School of Mathematics, University of Edinburgh, Edinburgh EH9 3JZ, United Kingdom

(Received 10 November 2005)

The dyadic Green function for a homogeneous electromagnetic medium inspired by the spatiotemporally nonhomogeneous constitutive equations of gravitationally affected vacuum is derived.

PACS: 41. 20. −q, 41. 20. Jb, 78. 20. −e

Vacuum or matter-free space is the most widely studied electromagnetic medium, not only because it underlies the development of continuum electromagnetic properties from microscopic principles,[1,2] but also because of the significance of electromagnetic communication devices in modern society.[3] The electromagnetic constitutive equations of vacuum are commonly stated in textbooks as

$$\boldsymbol{D}(\boldsymbol{r},t) = \epsilon_0 \boldsymbol{E}(\boldsymbol{r},t), \tag{1}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \boldsymbol{H}(\boldsymbol{r},t), \tag{2}$$

where $\epsilon_0 = 8.854 \times 10^{-12}\,\mathrm{F\ m^{-1}}$ and $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\ m^{-1}}$ in SI units, whereas $\boldsymbol{r}$ and t indicate position and time.

These equations presuppose either the absence of a gravitational field or that the observer is local. When a gravitational field is present, spacetime appears curved, which is well known.[4] One can still use the textbook versions of the Maxwell postulates for gravitationally affected vacuum, but the constitutive relations are now[5,6]

$$\boldsymbol{D}(\boldsymbol{r},t) = \epsilon_0 \underline{\underline{\gamma}}(\boldsymbol{r},t) \cdot \boldsymbol{E}(\boldsymbol{r},t) - c_0^{-1} \boldsymbol{\Gamma}(\boldsymbol{r},t) \times \boldsymbol{H}(\boldsymbol{r},t), \tag{3}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \underline{\underline{\gamma}}(\boldsymbol{r},t) \cdot \boldsymbol{H}(\boldsymbol{r},t) + c_0^{-1} \boldsymbol{\Gamma}(\boldsymbol{r},t) \times \boldsymbol{E}(\boldsymbol{r},t), \tag{4}$$

in lieu of Eqs. (1) and (2). Here $\underline{\underline{\gamma}}(\boldsymbol{r},t)$ is a real symmetric dyadic and $\boldsymbol{\Gamma}(\boldsymbol{r},t)$ is a vector with real-valued components, both related to the metric of spacetime; whereas $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$.

Just as isotropic dielectric-magnetic mediums provide material counterparts of Eqs. (1) and (2) in the frequency domain,[1,7] the vast variety of complex materials,[8,9] natural as well as artificial, suggests that it is quite possible that Eqs. (3) and (4) also have material counterparts. This thought inspired the present Letter, wherein we present the derivation of the dyadic Green function for frequency-domain electromagnetic fields in a homogeneous medium inspired by Eqs. (3) and (4).

With the assumption that all fields have an $\exp(-i\omega t)$ time dependence, with ω as the angular frequency, the constitutive relations of the chosen medium are

$$\boldsymbol{D}(\boldsymbol{r},\omega) = \epsilon_0 \underline{\underline{\gamma}}(\omega) \cdot \boldsymbol{E}(\boldsymbol{r},\omega) - c_0^{-1} \boldsymbol{\Gamma}(\omega) \times \boldsymbol{H}(\boldsymbol{r},\omega), \tag{5}$$

$$\boldsymbol{B}(\boldsymbol{r},\omega) = \mu_0 \underline{\underline{\gamma}}(\omega) \cdot \boldsymbol{H}(\boldsymbol{r},\omega) + c_0^{-1} \boldsymbol{\Gamma}(\omega) \times \boldsymbol{E}(\boldsymbol{r},\omega). \tag{6}$$

The coordinate system has been chosen such that $\underline{\underline{\gamma}}(\omega)$ is diagonal, and from now onwards the dependence on ω is implicit. Let us stress that Eqs. (5) and (6) are taken here to describe a material medium which can potentially be fabricated in a laboratory by properly dispersing electrically small bent-wire and other complex inclusions of different shapes and materials in some host materials,[10−13] but should not be confused with the constitutive equations (3) and (4) of gravitationally affected vacuum.

The frequency-domain Maxwell curl postulates in the chosen medium may be set down as

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = i\omega[\mu_0 \underline{\underline{\gamma}} \cdot \boldsymbol{H}(\boldsymbol{r}) + c_0^{-1} \boldsymbol{\Gamma} \times \boldsymbol{E}(\boldsymbol{r})], \tag{7}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = -i\omega[\epsilon_0 \underline{\underline{\gamma}} \cdot \boldsymbol{E}(\boldsymbol{r}) - c_0^{-1} \boldsymbol{\Gamma} \times \boldsymbol{H}(\boldsymbol{r})] + \boldsymbol{J}(\boldsymbol{r}), \tag{8}$$

where $\boldsymbol{J}(\boldsymbol{r})$ is the source electric current density. Our objective is to find the dyadic Green functions $\underline{\underline{G}}_e(\boldsymbol{r},\boldsymbol{s})$ and $\underline{\underline{G}}_m(\boldsymbol{r},\boldsymbol{s})$ such that

$$\boldsymbol{E}(\boldsymbol{r}) = i\omega\mu_0 \iiint \underline{\underline{G}}_e(\boldsymbol{r},\boldsymbol{s}) \cdot \boldsymbol{J}(\boldsymbol{s}) d^3\boldsymbol{s}, \tag{9}$$

$$\boldsymbol{H}(\boldsymbol{r}) = \iiint \underline{\underline{G}}_m(\boldsymbol{r},\boldsymbol{s}) \cdot \boldsymbol{J}(\boldsymbol{s}) d^3\boldsymbol{s}, \tag{10}$$

with the integrations being carried out over the region where the source electric current density is nonzero.

To begin with, the substitution of Eq. (9) into Eq. (7) and comparison of the resulting expression

* To whom correspondence should be addressed. Email: T.Mackay@ed.ac.uk

with Eq. (10) yield

$$\underline{\underline{G}}_m(\boldsymbol{r},\boldsymbol{s}) = \underline{\underline{\gamma}}^{-1}\cdot(\nabla\times\underline{\underline{I}} - ik_0\boldsymbol{\Gamma}\times\underline{\underline{I}})\cdot\underline{\underline{G}}_e(\boldsymbol{r},\boldsymbol{s}), \quad (11)$$

where $k_0 = \omega\sqrt{\epsilon_0\mu_0}$ and $\underline{\underline{I}}$ is the identity dyadic. Thus, an expression for only $\underline{\underline{G}}_e(\boldsymbol{r},\boldsymbol{s})$ has to be found.

For that purpose, following Lakhtakia and Welglhofer,[14] we start by defining new fields and source current density as

$$\boldsymbol{e}(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r})\exp(-ik_0\boldsymbol{\Gamma}\cdot\boldsymbol{r}), \quad (12)$$

$$\boldsymbol{h}(\boldsymbol{r}) = \boldsymbol{H}(\boldsymbol{r})\exp(-ik_0\boldsymbol{\Gamma}\cdot\boldsymbol{r}), \quad (13)$$

$$\boldsymbol{j}(\boldsymbol{r}) = \boldsymbol{J}(\boldsymbol{r})\exp(-ik_0\boldsymbol{\Gamma}\cdot\boldsymbol{r}). \quad (14)$$

Hence, Eqs. (7) and (8) respectively transform to

$$\nabla\times\boldsymbol{e}(\boldsymbol{r}) = i\omega\mu_0\underline{\underline{\gamma}}\cdot\boldsymbol{h}(\boldsymbol{r}), \quad (15)$$

$$\nabla\times\boldsymbol{h}(\boldsymbol{r}) = -i\omega\epsilon_0\underline{\underline{\gamma}}\cdot\boldsymbol{e}(\boldsymbol{r}) + \boldsymbol{j}(\boldsymbol{r}). \quad (16)$$

Next, we make use of an affine transformation associated with the scaling of space as per[15]

$$\tilde{\boldsymbol{r}} = \underline{\underline{\gamma}}^{1/2}\cdot\boldsymbol{r}, \quad (17)$$

where $\underline{\underline{\gamma}}^{1/2}\cdot\underline{\underline{\gamma}}^{1/2} = \underline{\underline{\gamma}}$ and we recall that $\underline{\underline{\gamma}}$ is a dyadic with real-valued elements. Let us define another set of fields and source current density as

$$\tilde{\boldsymbol{e}}(\boldsymbol{r}) = \underline{\underline{\gamma}}^{1/2}\cdot\boldsymbol{e}(\underline{\underline{\gamma}}^{1/2}\cdot\boldsymbol{r}), \quad (18)$$

$$\tilde{\boldsymbol{h}}(\boldsymbol{r}) = \underline{\underline{\gamma}}^{1/2}\cdot\boldsymbol{h}(\underline{\underline{\gamma}}^{1/2}\cdot\boldsymbol{r}), \quad (19)$$

$$\tilde{\boldsymbol{j}}(\boldsymbol{r}) = (\mathrm{adj}\underline{\underline{\gamma}}^{1/2})\cdot\boldsymbol{j}(\underline{\underline{\gamma}}^{1/2}\cdot\boldsymbol{r}), \quad (20)$$

where adj stands for the adjoint. Then, Eqs. (15) and (16) transform to

$$\nabla\times\tilde{\boldsymbol{e}}(\boldsymbol{r}) = i\omega\mu_0 g\tilde{\boldsymbol{h}}(\boldsymbol{r}), \quad (21)$$

$$\nabla\times\tilde{\boldsymbol{h}}(\boldsymbol{r}) = -i\omega\epsilon_0 g\tilde{\boldsymbol{e}}(\boldsymbol{r}) + \tilde{\boldsymbol{j}}(\boldsymbol{r}), \quad (22)$$

where

$$g = \sqrt{|\underline{\underline{\gamma}}|}, \quad (23)$$

and $|\underline{\underline{\gamma}}|$ denotes the determinant of $\underline{\underline{\gamma}}$.

From the foregoing equations, we obtain

$$[(\nabla\times\underline{\underline{I}})\cdot(\nabla\times\underline{\underline{I}}) - k_0^2g^2\underline{\underline{I}}]\cdot\nabla\times\tilde{\boldsymbol{e}}(\boldsymbol{r}) = i\omega\mu_0 g\tilde{\boldsymbol{j}}(\boldsymbol{r}). \quad (24)$$

The solution of Eq. (24) is well known as[16]

$$\tilde{\boldsymbol{e}}(\boldsymbol{r}) = i\omega\mu_0 g\iiint\tilde{\underline{\underline{g}}}(\boldsymbol{r},\boldsymbol{s})\cdot\tilde{\boldsymbol{j}}(\boldsymbol{s})d^3\boldsymbol{s}, \quad (25)$$

where

$$\tilde{\underline{\underline{g}}}(\boldsymbol{r},\boldsymbol{s}) = \Big(\underline{\underline{I}} + \frac{\nabla\nabla}{k_0^2g^2}\Big)\frac{\exp(ik_0g|\boldsymbol{r}-\boldsymbol{s}|)}{4\pi|\boldsymbol{r}-\boldsymbol{s}|}. \quad (26)$$

In order to go back from Eq. (25) to Eq. (9), we have to invert the two transformations in reverse sequence: Substituting Eqs. (18) and (20) into Eq. (25) yields

$$\boldsymbol{e}(\boldsymbol{r}) = i\omega\mu_0(\mathrm{adj}\underline{\underline{\gamma}}^{1/2})\cdot\Big(\iiint\underline{\underline{g}}(\boldsymbol{r},\boldsymbol{s})\cdot\boldsymbol{j}(\boldsymbol{s})d^3\boldsymbol{s}\Big), \quad (27)$$

wherein

$$\underline{\underline{g}}(\boldsymbol{r},\boldsymbol{s}) = \Big(\underline{\underline{I}} + \frac{1}{k_0^2g^2}\underline{\underline{\gamma}}\cdot\nabla\nabla\Big)\frac{\exp[ik_0g|\underline{\underline{\gamma}}^{-1/2}\cdot(\boldsymbol{r}-\boldsymbol{s})|]}{4\pi|\underline{\underline{\gamma}}^{-1/2}\cdot(\boldsymbol{r}-\boldsymbol{s})|}. \quad (28)$$

By substituting for $\boldsymbol{e}$ and $\boldsymbol{j}$ in Eq. (27) using Eqs. (12) and (14), respectively, we find

$$\boldsymbol{E}(\boldsymbol{r}) = i\omega\mu_0\exp(ik_0\boldsymbol{\Gamma}\cdot\boldsymbol{r})(\mathrm{adj}\underline{\underline{\gamma}}^{1/2})\cdot\Big(\iiint\underline{\underline{g}}(\boldsymbol{r},\boldsymbol{s})\cdot\boldsymbol{J}(\boldsymbol{s})\exp(-ik_0\boldsymbol{\Gamma}\cdot\boldsymbol{s})d^3\boldsymbol{s}\Big). \quad (29)$$

Therefore, the dyadic Green function $\underline{\underline{G}}_e(\boldsymbol{r},\boldsymbol{s})$ emerges from Eq. (29) as

$$\underline{\underline{G}}_e(\boldsymbol{r},\boldsymbol{s}) = \exp[ik_0\boldsymbol{\Gamma}\cdot(\boldsymbol{r}-\boldsymbol{s})](\mathrm{adj}\underline{\underline{\gamma}}^{1/2})\cdot\Big(\underline{\underline{I}} + \frac{1}{k_0^2g^2}\underline{\underline{\gamma}}\cdot\nabla\nabla\Big)\frac{\exp[ik_0g|\underline{\underline{\gamma}}^{-1/2}\cdot(\boldsymbol{r}-\boldsymbol{s})|]}{4\pi|\underline{\underline{\gamma}}^{-1/2}\cdot(\boldsymbol{r}-\boldsymbol{s})|}. \quad (30)$$

Equation (30) is the desired result. If $\underline{\underline{\gamma}} = \underline{\underline{I}}$ and $\boldsymbol{\Gamma} = \boldsymbol{0}$, this expression reduces to the usual dyadic Green function for gravitationally unaffected vacuum.[16]

References

[1] Jackson J D 1999 *Classical Electrodynamics* 3rd edn (New York: Wiley) chap 6

[2] Lakhtakia A and Messier R 2005 *Sculptured Thin Films: Nanoengineered Morphology and Optics* (Bellingham, WA: SPIE) chap 6

[3] Kraus J D 1984 *Electromagnetics* 3rd edn (New York: McGraw–Hill)

[4] Schutz B F 1985 *A First Course in General Relativity* (Cambridge: Cambridge University Press)

[5] Plebanski J 1960 *Phys. Rev.* **118** 1396

[6] Mackay T G, Lakhtakia A and Setiawan S 2005 *New J. Phys.* **7** 75

[7] Elliott R S 1993 *Electromagnetics: History, Theory, and Applications* (New York: IEEE)

[8] Gersten J I and Smith F W 2001 *The Physics and Chemistry of Materials* (New York: Wiley)

[9] Weiglhofer W S and Lakhtakia A (eds) 2003 *Introduction to Complex Mediums for Optics and Electromagnetics* (Bellingham, WA: SPIE)

[10] Marqués R, Medina F and Rafii-El-Idrissi R 2002 *Phys. Rev.* B **65** 144440

[11] Mackay T G and Lakhtakia A 2004 *Phys. Rev.* E **69** 026602

[12] Fiebig M 2005 *J. Phys.* D **38** R123

[13] Gich M, Frontera C, Roig A, Fontcuberta J, Molins E, Bellido N, Simon Ch and Fleta C 2006 *Nanotechnology* **17** 687

[14] Lakhtakia A and Weiglhofer W S 1997 *Microw. Opt. Technol. Lett.* **15** 168

[15] Lakhtakia A and Weiglhofer W S 1994 *Int. J. Appl. Electromag. Mater.* **5** 101

[16] Chen H C 1983 *Theory of Electromagnetic Waves* (New York: McGraw–Hill) chap 9