Variational Approach to Study $\mathcal{PT}$-Symmetric Solitons in a Bose–Einstein Condensate with Non-locality of Interactions *

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Considering the non-locality of interactions in a Bose–Einstein condensate, the existence and stability of solitons subject to a $\mathcal{PT}$-symmetric potential are discussed. In the framework of the variational approach, we investigate how the non-locality of interactions affects the self-localization and stability of a condensate with attractive two-body interactions. The results reveal that the non-locality of interactions dramatically influences the shape, width, and chemical potential of the condensate. Analytically variational computation also predicts that there exists a critical negative non-local interaction strength ($p_c < 0$) with each fixed two-body interaction ($g_0 < 0$), and there exists no bright soliton solution for $p_0 < p_c$. Furthermore, we study the effect of the non-locality interactions on the stability of the solitons using the Vakhitov–Kolokolov stability criterion. It is shown that for a positive non-local interaction ($p_0 > 0$), there always exist stable bright solitons in some appropriate parameter regimes.

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Solitons are solitary and shape preserving traveling waveforms, which commonly exist in nonlinear media. The ultra-cold atomic Bose–Einstein condensates (BECs) are described by the non-linear Gross–Pitaevskii (GP) equation which exhibits such soliton solutions. In this system, the sign of the nonlinearity depends on the scattering length of the atoms. One can obtain a bright soliton solution in an attractive BEC with negative s-wave scattering length, and a dark soliton solution in a repulsive BEC with positive s-wave scattering length.

In ultra-cold atomic systems, interaction with the environment often plays an important role, leading to a gain or loss of particles. In the mean-field approximation of the GP equation, the necessary particle gain and loss can be described by imaginary potentials, rendering the Hamiltonian non-Hermitian.[1,2] The particles' in- and out-coupling were compared to many-particle calculations justifying their use in mean-field theory.[3,4] In 1998, Bender and Boettcher[5] discovered that non-Hermitian Hamiltonians can support stationary solutions if they are $\mathcal{PT}$-symmetric. Based on this, many interesting nonlinear phenomena have been recovered in BECs and related fields. For example, nonlinear $\mathcal{PT}$-symmetric quantum systems have been discussed for BECs described in a two-mode approximation.[6,7] Researchers have investigated nonlinear quantum dynamics in a $\mathcal{PT}$ double well,[8] and vortices in BECs with a $\mathcal{PT}$-symmetric potential,[9] rogue waves in the (2+1)-dimensional nonlinear Schrödinger equation with a $\mathcal{PT}$-symmetric potential,[10] and transverse localization of light in 1D self-focusing $\mathcal{PT}$-symmetric optical lattices.[11]

The GP equation used to describe BECs considers s-wave scattering between bosons. This shape-independent interaction approximation assumes that the gas density n_0 multiplying s-wave scattering length a satisfies $n_0 a \ll 1$.[12] However, using Feshbach resonance technology to tune the s-wave interactions, the parameter $n_0 a$ can reach 0.05 or higher.[13,14] In this regime, the shape dependence of the s-wave interactions is expected to come into play. The modified GP equation that includes non-locality interaction has been introduced[15−18] to account for the shape dependence of the inter-boson interactions. Including this non-locality interaction, many interesting phenomena have been revealed in a real potential. The modulational stability condition is dramatically influenced by non-locality interactions.[19] With the help of the non-locality interaction, a usually unstable BEC with a negative scattering length can be stabilized by positive higher-order effects.[20] The exact solutions of the modified GP equation describing bright and dark solitons have been studied.[12]

In this Letter, combining these two hot topics, we focus on how the non-local nonlinearity has an effect on the existence and stability properties of bright solitons trapped in a $\mathcal{PT}$-symmetric potential. Using the variational approach, we analytically obtain an expression to investigate how the interplay between $\mathcal{PT}$-symmetry and non-locality interaction influences the soliton properties. Meanwhile, we comb the Vakhitov–

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Kolokolov (VK) stability condition, i.e., the VK condition determines the stability of the solitons based on the sign of the derivative of the momentum versus speed graph for dark solitons, and on the particle number versus chemical potential graph for bright solitons. The results show that a relatively large negative non-locality interaction can destroy a bright soliton, and a positive non-local interaction dramatically changes the width, chemical potential and stability properties of the bright soliton.

To obtain a model explaining the non-locality of the s-wave interactions, we consider an additional term to the local GP equation, which is referred to as the modified GP equation[15–18] given by

$$i\hbar\frac{\partial \Psi}{\partial t}=\Big[-\frac{\hbar^2}{2m}\nabla^2+V(\boldsymbol{r})+g_0|\Psi|^2+p_0\nabla^2|\Psi|^2\Big]\Psi, \tag{1}$$

where $\hbar$ is the reduced Planck constant, m is the mass of a single boson, $g_0=4\pi\hbar^2 a/m$ is the strength of two-body contact interactions, with the s-wave scattering length a corresponding to the lowest-order zero-range potential of the two-body contact interactions, $p_0=g_0(\frac{1}{3}a^2-\frac{1}{2}r_{\rm e})$ with $r_{\rm e}$ being the effective range, and p_0 incorporates the shape-dependence on the interaction potential through $r_{\rm e}$, and then accounts for the non-local interaction. Assuming that the condensate is harmonically trapped in the y and z directions and with an axial trap potential $V(x)$ in the x direction, the external trapping potential can be written as $V(\boldsymbol{r})=\frac{1}{2}m\omega_\perp(y^2+z^2)+V(x)$. The radial motion can be strongly confined by making the radial trapping frequency $\omega_\perp\gg\omega_x$. In this case the condensate is cigar-shaped, so one can take $\Psi(\boldsymbol{r},t)=\phi_0\psi(x,t)$, where $\phi_0=\sqrt{\frac{1}{\pi a_\perp^2}}\exp(-\frac{r_\perp^2}{2a_\perp^2})$ is the ground state of the radial problem, with $a_\perp=\sqrt{\hbar/(m\omega_\perp)}$ and $r_\perp=\sqrt{y^2+z^2}$ the radial distance. Then multiplying both sides of the modified GP equation (1) by ϕ_0^* and integrating over the transverse variable $r_\perp$, we obtain a quasi-one-dimensional GP equation in the form

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V(x)\psi+g_0|\psi|^2\psi+p_0\frac{\partial|\psi|^2}{\partial x^2}\psi. \tag{2}$$

We rescale the wave function, space and time as follows:[21] $\tau=\frac{1}{2}\omega_\perp t$, $\tilde{x}=a_\perp^{-1}x$, $\tilde{\psi}=\sqrt{a_\perp^{-3}}\psi$, $\tilde{g_0}=\frac{2a_\perp^3}{\hbar\omega_\perp}g_0$, $\tilde{p_0}=\frac{2a_\perp}{\hbar\omega_\perp}p_0$, and $\tilde{V}(x)=V(x)/\hbar\omega_\perp$. We will continue to use t instead of τ, and we drop the tildes for simplicity. The dynamics of the cigar-shaped condensate can be governed by a quasi-one-dimensional GP equation, which has the dimensional form

$$i\frac{\partial\psi}{\partial t}=-\frac{\partial^2\psi}{\partial x^2}+V(x)\psi+g_0|\psi|^2\psi+p_0\frac{\partial|\psi|^2}{\partial x^2}\psi, \tag{3}$$

where we take $V(x)$ to be complex valued and $\mathcal{PT}$-symmetric, i.e., possessing a symmetric real part and an antisymmetric imaginary part. The potential has the form $V(x)=V_{\rm R}(x)+iV_{\rm I}(x)$. We choose a Gaussian-shaped potential $V_{\rm R}=-V_{\rm r}\exp(-x^2)$ as the real part of the $\mathcal{PT}$ potential. Such a trap can be implemented using optical dipole traps. The imaginary part is chosen as a Gaussian multiplied by x, expressed as $V_{\rm I}=-V_{\rm i}x\exp(-x^2)$. Figure 1 depicts the real part and the imaginary part of $V(x)$, and the imaginary part represents the gain-loss mechanics.

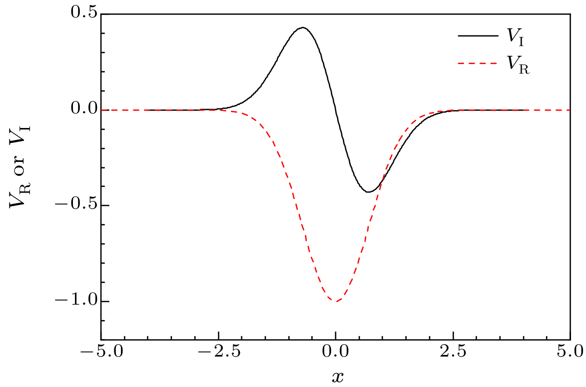

Fig. 1. The transverse profiles of the real and imaginary parts of the potential $V(x)$. The black solid line represents the imaginary part, and the red dotted line the real part, with $V_{\rm r}=V_{\rm i}=1$.

We are interested in devising a variational principle for obtaining the equation of the wave function for the GP equation with a complex potential. However, the complexity of the potential makes the problem non-conservative. Here we use the variational approach developed for dissipative systems. Inserting the ansatz $\psi=\phi(x)\exp(-i\mu t)$ into Eq. (3), we obtain

$$\mu\phi+\frac{d^2\phi}{dx^2}-g_0|\phi|^2\phi-p_0\frac{\partial|\phi|^2}{\partial x^2}\phi+V_{\rm r}\exp(-x^2)\phi$$
$$=iV_{\rm I}(x)\phi. \tag{4}$$

The Lagrangian for the conservative part, corresponding to the left-hand side of Eq. (4), is

$$L_{\rm c}=\frac{i}{2}\Big(\frac{\partial\phi^*}{\partial t}\phi-\frac{\partial\phi}{\partial t}\phi^*\Big)-\mu|\phi|^2+\Big|\frac{\partial\phi}{\partial x}\Big|^2$$
$$-V_{\rm r}\exp(-x^2)|\phi|^2+\frac{g_0}{2}|\phi|^4+\frac{1}{4}p_0|\phi|^2\frac{\partial^2|\phi|^2}{\partial x^2}. \tag{5}$$

To apply the variational approach, we choose a trial solution with the nonzero phase associated with $\mathcal{PT}$-symmetric stationary states,

$$\psi=A\exp\Big(-\frac{x^2}{\omega_{\rm b}^2}\Big)\exp[i\theta f(x)], \tag{6}$$

where A corresponds to the condensate amplitude supposed to be real, $\omega_{\rm b}$ is the width of the soliton, θ is the amplitude of the phase profile, and $f(x)$ is the phase distribution along x. The particle number of the solution is defined as $N=\int_{-\infty}^{\infty}|\phi|^2dx=\sqrt{\frac{\pi}{2}}A^2\omega_{\rm b}$. Inserting Eq. (6) into Eq. (5), we obtain the following

Lagrangian

$$\begin{aligned}L_c =& -\mu A^2 \exp\Big(-\frac{2x^2}{\omega_b^2}\Big)\\ &+ A^2 \exp\Big(-\frac{2x^2}{\omega_b^2}\Big)\theta^2\Big[\frac{df(x)}{dx}\Big]^2\\ &+\Big(\frac{2x^2}{\omega_b^2}\Big)^2 A^2 \exp\Big(-\frac{2x^2}{\omega_b^2}\Big)\\ &- V_r \exp(-x^2) A^2 \exp\Big(-\frac{2x^2}{\omega_b^2}\Big)\\ &+\frac{g_0}{2}A^4\exp\Big(-\frac{4x^2}{\omega_b^2}\Big)\\ &+p_0 A^4 \exp\Big(-\frac{4x^2}{\omega_b^2}\Big)\Big(\frac{4x^2}{\omega_b^4}-\frac{1}{\omega_b^2}\Big).\end{aligned}\tag{7}$$

From Eq. (7) we obtain the following reduced Lagrangian $\langle L_c\rangle = \int_{-\infty}^{\infty} L_c dx$,

$$\begin{aligned}\langle L_c\rangle =& \frac{-\mu A^2\sqrt{\pi}}{\sqrt{2}} + A^2\theta^2 \int_{-\infty}^{\infty}\Big[\frac{df(x)}{dx}\Big]^2\\ &\cdot \exp\Big(-\frac{2x^2}{\omega_b^2}\Big)dx + \frac{1}{\omega_b^4}\frac{A^2\sqrt{\pi}\omega_b^3}{\sqrt{2}}\\ &- V_r A^2 \int_{-\infty}^{\infty}\exp(-x^2)\exp\Big(-\frac{2x^2}{\omega_b^2}\Big)dx\\ &+ g_0\frac{A^4}{2}\frac{\sqrt{\pi}}{2}\omega_b - \frac{A^4\sqrt{\pi}p_0}{4\omega_b},\end{aligned}\tag{8}$$

which can be further simplified by rewriting in terms of particle number N, obtaining

$$\begin{aligned}\langle L_c\rangle =& -\mu N + \frac{N}{\omega_b}\sqrt{\frac{2}{\pi}}\theta^2\int_{-\infty}^{\infty}\Big[\frac{df(x)}{dx}\Big]^2\\ &\cdot\exp\Big(-\frac{2x^2}{\omega_b^2}\Big)dx + \frac{N}{\omega_b^2}\\ &- \frac{\sqrt{2}N}{\sqrt{\pi}}\frac{V_r}{\omega_b}\int_{-\infty}^{\infty}\exp(-x^2)\exp\Big(-\frac{2x^2}{\omega_b^2}\Big)dx\\ &+\frac{g_0 N^2}{2\sqrt{\pi}\omega_b} - \frac{p_0 N^2}{2\sqrt{\pi}\omega_b^3},\end{aligned}\tag{9}$$

where the last two terms on the right-hand side depend on g_0 and p_0, stemming from the two-body interaction and the non-locality interaction, respectively. Using the standard variational approach, the systems with dissipative terms can be modified as[22,23]

$$\frac{d}{dt}\Big(\frac{\partial\langle L_c\rangle}{\partial\varphi_t}\Big) - \frac{\partial\langle L_c\rangle}{\partial\varphi} = 2\mathrm{Re}\int_{-\infty}^{\infty} Q\frac{\partial\phi^*}{\partial\varphi}dx,\tag{10}$$

where $\varphi = N, \omega_b, \theta$, and $Q = iV_I(x)\phi$ representing the gain-loss of a particle from the environment.

Choosing $\varphi = N$, from Eq. (10), we can obtain an explicit expression for the chemical potential μ as

$$\begin{aligned}\mu =& \frac{\sqrt{2}\theta^2}{\sqrt{\pi}\omega_b}\int_{-\infty}^{\infty}\Big[\frac{df(x)}{dx}\Big]^2\exp\Big(-\frac{2x^2}{\omega_b^2}\Big)dx\\ &+\frac{1}{\omega_b^2}+\frac{g_0N}{\sqrt{\pi}\omega_b}-\frac{p_0N}{\sqrt{\pi}\omega_b^3}\\ &-\frac{\sqrt{2}V_r}{\sqrt{\pi}\omega_b}\int_{-\infty}^{\infty}\exp(-x^2)\exp\Big(-\frac{2x^2}{\omega_b^2}\Big)dx,\end{aligned}\tag{11}$$

where the first term on the right-hand side stems from the influence of the phase profile $\theta f(x)$ on the nonlinear eigenvalue μ. The second term stems from the dispersive spreading. The third and forth terms originate from the two-body interaction and non-locality interaction terms, respectively. It is clearly shown that they have a competitive relationship on chemical potential, due to the fact that the signs of the third term and the fourth term are opposite. Therefore, the influences of two-body interactions and non-locality interactions on the chemical potential μ are different intuitively. The last term accounts for the linear trapping potential V_R.

Secondly, when we choose $\varphi = \omega_b$, from Eq. (10) we obtain

$$\begin{aligned}&\frac{1}{\omega_b^3}+\frac{g_0N}{4\sqrt{\pi}\omega_b^2}-\frac{3p_0N}{4\sqrt{\pi}\omega_b^4}\\ =&\frac{\theta^2}{\sqrt{2}\sqrt{\pi}}\frac{\partial}{\partial\omega_b}\Big\{\frac{1}{\omega_b}\int_{-\infty}^{\infty}\Big[\frac{df(x)}{dx}\Big]^2\exp\Big(-\frac{2x^2}{\omega_b^2}\Big)dx\Big\}\\ &-\frac{V_r}{\sqrt{2}\sqrt{\pi}}\frac{\partial}{\partial\omega_b}\Big\{\frac{1}{\omega_b}\int_{-\infty}^{\infty}\exp(-x^2)\\ &\cdot\exp\Big(-\frac{2x^2}{\omega_b^2}\Big)dx\Big\},\end{aligned}\tag{12}$$

where the left-hand side states the competition among the dispersion of the particles (first term), two-body interaction (second term), and non-locality interaction (third term) to determine the condensate width if the trapping potential is absent. The first term on the right-hand side arises from the inhomogeneous phase of the soliton, and the second term accounts for the influence of the linear trapping potential on the condensate width.

Finally, the equation for θ is obtained by choosing $\varphi = \theta$, and from this we obtain

$$\theta = -\frac{\int_{-\infty}^{\infty} V_I(x)\exp(-\frac{2x^2}{\omega_b^2})f(x)dx}{\int_{-\infty}^{\infty}[\frac{df(x)}{dx}]^2\exp(-\frac{2x^2}{\omega_b^2})dx},\tag{13}$$

where the phase of the solution is immune from atomic interaction. For purely real potentials ($V_I = 0$), stationary solutions feature a flat phase profile across x.

In the following, using the variational results, we focus on how the non-locality interaction influences the existence and stability of a soliton. In our variational analysis, we reasonably choose $f(x) = \tanh(x)$

in Eq. (6) as discussed in Ref. [24]. From Eqs. (11)–(13), the chemical potential μ, particle number N and phase profile θ of the soliton can be obtained. We find that the non-locality interaction dramatically influences the soliton existence and stability properties. Mainly to discuss the effects of the non-locality interactions on the properties of a bright soliton, we vary the non-locality interaction parameter p_0, and fix the attractive two-body interaction $g_0 = -1$ in the following. Figures 2(a1)-2(a3) show the influence of a negative p_0 on the chemical potential μ, soliton width $\omega_{\rm b}$, and the corresponding soliton profile. We find that there exists a critical value of $p_{\rm c}$, where $p_{\rm c} < 0$. In the case of $p_0 < p_{\rm c}$, there is no bright soliton to survive. In other words, if the absolute value of the negative non-locality interaction is larger, the non-locality interaction can destroy the bright soliton. However, if the absolute value of the negative non-locality interaction is small, it only slightly influences the soliton width and chemical potential for a fixed particle number N as shown in Figs. 2(a1) and 2(a2). Meanwhile, we find that when the particle number N is small, the impact of the non-locality interaction on μ and $\omega_{\rm b}$ can be negligible. However, with an increase of N, the effects from non-locality interaction are amplified, and the soliton width $\omega_{\rm b}$ and μ grow with the absolute value of p_0. Figure 2(a3) shows the soliton profile, and the corresponding parameter is marked in Fig. 2(a1) with circles. It is shown that in the case of $\mu = -20$, following the non-locality interaction strength from -0.01 to 0, the amplitude of the density profile becomes lower and lower.

Figures 2(b1) and 2(b2) show the relationship between the chemical potential μ and width $\omega_{\rm b}$ versus the particle number N for $p_0 > 0$. It is clearly shown that a positive p_0 dramatically changes the growth tendency between μ and width $\omega_{\rm b}$ versus the particle number N. The relationship between μ and $\omega_{\rm b}$ no longer monotonically changes with N. In the regime where $d\omega_{\rm b}/dN$ or $d\mu/dN$ is negative, the value of soliton width $\omega_{\rm b}$ or the value of μ declines with increasing particle number N. However, in the regime where $d\omega_{\rm b}/dN$ or $d\mu/dN$ is positive, with the increase of N, the values of $\omega_{\rm b}$ or the values of μ all increase. Figure 2(b3) shows the density profile of solitons; the corresponding parameter is marked in Fig. 2(b1), where the chemical potential $\mu = -4.18$. Figure 2(b3) shows that, with the increase of p_0, the amplitude of the soliton decreases.

Note that the stability of localized modes in nonlinear systems is very important. Therefore, in this part, we focus on the stability of $\mathcal{PT}$-symmetric solitons. The VK stability condition ensures the stability of solitons for the 1D local GP equation.[25] The VK condition determines the stability of the solitons based on the sign of particle number versus chemical potential graph for bright solitons. If $dN/d\mu$ is positive, the corresponding soliton will be stable, otherwise the soliton will be unstable. Recently, the VK condition has also been extended to nonlinear Schrödinger equations in complex external potentials to predict the domain of stability.[26–28] From Fig. 2(a1), it is obviously shown that if the non-locality interaction strength satisfies the condition $p_0 \leq 0$, for a negative g_0, the $\mathcal{PT}$-symmetric solitons will always be unstable. These results also coincide with the numerical results in Ref. [23]. Due to the positive non-local interaction (p_0), it can dramatically change the sign of $dN/d\mu$ for different parameter regimes. As shown in Fig. 2(b1) with different positive values of p_0, in the regime of $dN/d\mu > 0$, the corresponding solitons are always stable. The stable solitons as marked in Fig. 2(b1) are plotted in Fig. 2(b3) with solid and dotted lines. For the case of $dN/d\mu < 0$, in this parameter regime the corresponding solitons are unstable. In the case under consideration, the unstable soliton density profile is shown in Fig. 2(b3) with a dashed line.

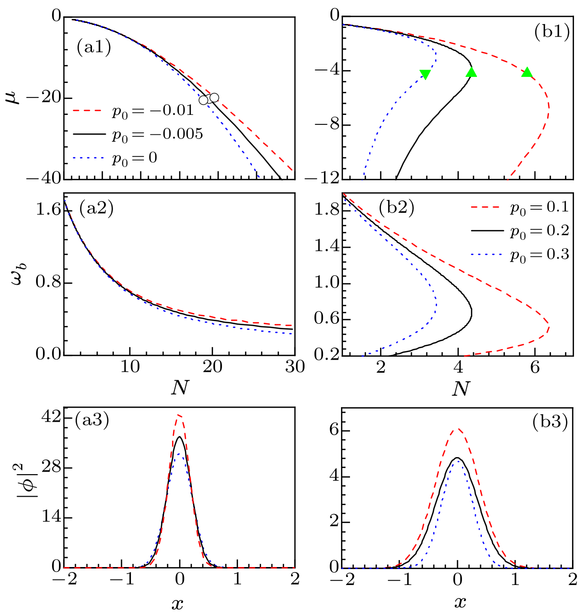

Fig. 2. Behavior of μ (a1) and the soliton width $\omega_{\rm b}$ (a2) for a negative p_0 versus the particle number N. The soliton profile with the parameter marked in (a1) versus x is presented in (a3). In all the cases, $g_0 = -1$. The influence of positive p_0 on μ (b1), the soliton width $\omega_{\rm b}$ (b2), and the corresponding soliton profile marked in (b1) is plotted in (b3), with the parameter $g_0 = -1$.

In summary, we have studied the existence and stability of localized states of a BEC with non-locality interaction trapped in a $\mathcal{PT}$-symmetric potential. Within the mean-field framework, we find that the interplay between non-locality interaction and gain-loss mechanics dramatically influences the soliton state in a BEC. The non-locality interaction can drastically change the $\mathcal{PT}$-symmetric soliton's profile, chemical potential and width. Most importantly, we find that a negative non-locality interaction can destroy a bright soliton. With the help of a positive non-locality interaction p_0, there will exist stable bright $\mathcal{PT}$-symmetric solitons in a BEC with attractive two-body interactions.

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