CHIN. PHYS. LETT. Vol. 34, No. 11 (2017) 110301

Non-Relativistic Treatment of a Generalized Inverse Quadratic Yukawa Potential

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(Received 16 August 2017)

A bound state solution is a quantum state solution of a particle subjected to a potential such that the particle's energy is less than the potential at both negative and positive infinity. The particle's energy may also be negative as the potential approaches zero at infinity. It is characterized by the discretized eigenvalues and eigenfunctions, which contain all the necessary information regarding the quantum systems under consideration. The bound state problems need to be extended using a more precise method and approximation scheme. This study focuses on the non-relativistic bound state solutions to the generalized inverse quadratic Yukawa potential. The expression for the non-relativistic energy eigenvalues and radial eigenfunctions are derived using proper quantization rule and formula method, respectively. The results reveal that both the ground and first excited energy eigenvalues depend largely on the angular momentum numbers, screening parameters, reduced mass, and the potential depth. The energy eigenvalues, angular momentum numbers, screening parameters, reduced mass, and the potential depth or potential coupling strength determine the nature of bound state of quantum particles. The explored model is also suitable for explaining both the bound and continuum states of quantum systems.

PACS: 03.65.−w, 03.65.Ca, 03.65.Ge DOI: 10.1088/0256-307X/34/11/110301

The dynamics and interactions of non-relativistic particles (atoms and molecules) can usually be described in terms of a physical potential model using the time-independent Schrödinger wave equation.[1–10] The analytical solutions to this equation with physically solvable potentials play an important role in our understanding of the physical background of a quantum system. This is attributed to the fact that the wave functions associated with quantum problems contain all the necessary information regarding the quantum systems under consideration. The Schrödinger equation is the second-order homogeneous differential equation which can be used to determine the possible wavefunctions and the energies of some non-relativistic particles.

The Schrödinger equation can be generalized to three dimensions and the corresponding wave functions are functions of three-dimensional space coordinates. The impressive and overwhelming success of solving the Schrödinger equation is that of the prediction of energy levels, eigenfunctions and the transition probabilities, which is forbidden by Newtonian mechanics. Consequent upon these, the exact and approximate solutions to the Schrödinger equation with diverse potential models have been obtained in no small measure using different types of approximations and methods (see Refs. [10–24] and other references therein).

The aim of this study is to obtain the approximate bound state solutions to the Schrödinger equation with the generalized inverse quadratic Yukawa potential. The proper quantization rule and formula method were used to obtain the energy eigenvalue and the radial eigenfunction within a short range approximation so as to overcome the effect of the centrifugal barrier term.

In this work, the proper quantization rule would be employed to obtain the bound state energy. This method finds its applications in various fields of physics including mathematical, nuclear, atom, chemical and molecular physics. The proper quantization rule is a simple, powerful and most suitable tool for obtaining bound state energy spectra of physically solvable potentials. Several works have successfully applied this method to bound states problems.[25–35] The detailed precepts of this method are strictly followed here to obtain the bound state energy of the generalized inverse quadratic Yukawa potential model.

Since the proper quantization rule cannot be used to calculate the wave functions, we employ the formula method suggested by Falaye *et al.* because of its simplicity and efficiency. This method can also be used to obtain the energy level of quantum systems.[36] The wave equations with any physical potential model of interest are transformed into the Schrödinger-like equation as follows:

$$\psi''(z)+\frac{(k_1-k_2z)}{z(1-k_3z)}\psi'(z)+\frac{(Az^2+Bz+C)}{z^2(1-k_3z)^2}\psi(z)=0. \tag{1}$$

Using the coordinate transformation variable of $z=z(r)$, we obtain the energy eigenvalues and the corresponding wave function as

$$k_5^2-\Big\{\Big\{k_4^2-k_5^2-\Big[\frac{1-2n}{2}-\frac{1}{2k_3}(k_2-\sqrt{(k_3-k_2)^2-4A}\Big]^2\Big\}\Big/\Big\{2\Big[\frac{1-2n}{2}-\frac{1}{2k_3}(k_2-\sqrt{(k_3-k_2)^2-4A}\Big]\Big\}\Big\}^2=0,\ k_3\neq 0, \tag{2}$$

$$\psi(z)=N_{n,l}\ z^{k_4}(1-k_3)^{k_5}{}_2F_1\Big(-n,n+2(k_4+k_5)+\frac{k_2}{k_3}-1;2k_4+k_1,k_3z\Big), \tag{3}$$

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where

$$k_4 = \frac{(1-k_1)+\sqrt{(1-k_1)^2-4C}}{2}, \tag{4}$$

$$k_5 = \frac{1}{2}+\frac{k_1}{2}-\frac{k_2}{2k_3} + \sqrt{\Big[\frac{1}{2}+\frac{k_1}{2}-\frac{k_2}{2k_3}\Big]^2-\Big[\frac{A}{k_3^2}+\frac{B}{k_3}+C\Big]}. \tag{5}$$

Note that $N_{n,l}$ and ${}_2F_1$ are the normalization constant and the hypergeometric function, respectively.

The generalized inverse quadratic Yukawa potential (GIQYP) is written as[37]

$$\begin{aligned} V(r) &= -V_0\Big(1+\frac{1}{r}e^{-\alpha r}\Big)^2 \\ &= -\frac{A'}{r^2}e^{-2\alpha r}-\frac{B'}{r}e^{-\alpha r}-C', \\ A' &= C' = V_0,\ B' = 2V_0, \end{aligned} \tag{6}$$

where α is the screening parameter, and V_0 is the coupling strength of the potential. This potential is a combination of the inverse quadratic Yukawa (IQY) and the Yukawa potential. It is asymptotic to a finite value as $r \to \infty$ and becomes infinite at $r = 0$.

The radial Schrödinger equation with this potential can be expressed as

$$\frac{d^2U(r)}{dr^2}+\frac{2\mu}{\hbar^2}[E-V_{\rm eff}]U(r) = 0, \tag{7}$$

with the effective potential

$$V_{\rm eff} = V(r)+\frac{l(l+1)\hbar^2}{2\mu r^2}. \tag{8}$$

Putting Eqs. (6) and (8) into Eq. (7), we obtain

$$\frac{d^2U(r)}{dr^2}+\frac{2\mu}{\hbar^2}[E+C'+\frac{A'}{r^2}e^{-2\alpha r} + \frac{B'}{r}e^{-\alpha r}-\frac{l(l+1)\hbar^2}{2\mu r^2}]U(r) = 0. \tag{9}$$

To obtain bound state solutions to Eq. (9), we employ the proper quantization rule. Using the Pekeris-type approximation to deal with the centrifugal term[38,39]

$$\frac{1}{r^2} \approx 4\alpha^2\frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}, \tag{10}$$

and introducing a new variable $y = (e^{2\alpha r}-1)^{-1}$, we have

$$\frac{d^2U(y)}{dy^2}+\frac{2\mu}{\hbar^2}[E-V_{\rm eff}(y)]U(y) = 0, \tag{11}$$

where

$$\begin{aligned} V_{\rm eff}(y) &= P+Qy+Ry^2,\quad P = -C', \\ Q &= 4\alpha^2 l(l+1)\frac{\hbar^2}{2\mu}-2\alpha B', \\ R &= 4\alpha^2 l(l+1)\frac{\hbar^2}{2\mu}-4\alpha A'. \end{aligned} \tag{12}$$

The two turning points $y_{\rm A}$ and $y_{\rm B}$ are obtained by solving quadratically for $V_{\rm eff}(y)-E_{nl} = 0$

$$\begin{aligned} y_{\rm A} &= -\frac{Q}{2R}-\frac{1}{2R}\sqrt{Q^2-4R(P-E)}, \\ y_{\rm B} &= -\frac{Q}{2R}+\frac{1}{2R}\sqrt{Q^2-4R(P-E)}, \end{aligned} \tag{13}$$

with the properties

$$y_{\rm A}+y_{\rm B} = -\frac{Q}{R},\quad y_A y_{\rm B} = \frac{P-E}{R}. \tag{14}$$

The momentum $k(y)$ between the two turning points $y_{\rm A}$ and $y_{\rm B}$ can be evaluated as

$$\begin{aligned} k(y) &= \sqrt{\frac{2\mu R}{\hbar^2}}\Big\{(-1)\Big[\frac{(P-E)}{R}+\frac{Q}{R}y+y^2\Big]\Big\}^{1/2} \\ &= \sqrt{\frac{2\mu R}{\hbar^2}}[(-1)(y-y_{\rm A})(y-y_{\rm B})]^2. \end{aligned} \tag{15}$$

Equation (7) can also be written in the form of the Riccati equation as

$$-2\alpha y(1+y)\phi_0'(y)+[\phi_0(y)]^2 = -\frac{2\mu}{\hbar^2}[E_0-P-Qy-Ry^2]. \tag{16}$$

Considering the monotonic property, the logarithmic derivative form $\phi_0(y)$ for the ground state has one node and no pole, thus we have to take a linear form in y. Therefore, we assume a trial solution of the form $\phi_0(y) = A+By$, and substituting it into Eq. (16) we obtain

$$\begin{aligned} &A^2+(-2\alpha B+2AB)y+(-2\alpha B+B^2)y^2 \\ &= -\frac{2\mu}{\hbar^2}(E_0-P)+\frac{2\mu Q}{\hbar^2}y+\frac{2\mu R}{\hbar^2}y^2. \end{aligned} \tag{17}$$

Comparing the LHS and RHS of Eq. (17) and solving for A and B quadratically, we obtain the values of A and B as

$$\begin{aligned} A &= \sqrt{-\frac{2\mu}{\hbar^2}(E_0-P)}, \\ B &= \Big[\alpha\pm\sqrt{\alpha^2+\frac{2\mu R}{\hbar^2}}\Big], \end{aligned} \tag{18}$$

while the problem can only be physically solvable for $B = [\alpha+\sqrt{\alpha^2+\frac{2\mu R}{\hbar}}^2]$. Thus the integral of the momentum $k(r)$ in terms of y is

$$\int_{r_{\rm A}}^{r_{\rm B}} k(r)dr = -\int_{y_{\rm A}}^{y_{\rm B}}\Big\{\frac{1}{2\alpha}\sqrt{\frac{2\mu R}{\hbar^2}} \cdot\Big[\frac{\sqrt{(-1)(y-y_{\rm A})(y-y_{\rm B})}}{y(1+y)}\Big]\Big\}dy. \tag{19}$$

To obtain the energy levels of all the bound states, we consider the following useful integrals[31]

$$\int_{y_A}^{y_B} \left\{ \left[\frac{\sqrt{(-1)(y-y_A)(y-y_B)}}{y(1+y)} \right] \right\} dy = \pi[\sqrt{(y_A+1)(y_B+1)} - 1 - \sqrt{y_A y_B}]. \quad (20)$$

Substituting Eq. (20) into Eq. (19), we now evaluate the integral of the momentum $k(r)$ as

$$\int_{r_A}^{r_B} k(r)dr = -\frac{\pi}{2\alpha}\sqrt{\frac{2\mu R}{\hbar^2}}\left[\left(\frac{R-Q+P-E_0}{R}\right)^{1/2} - 1 - \left(\frac{P-E_0}{R}\right)^{1/2}\right]. \quad (21)$$

With the aid of Eqs. (17) and (18) and following algebraic means, we have

$$\frac{\hbar^2}{2\mu R}(A-B)^2 = \frac{R-Q+P-E_0}{R}. \quad (22)$$

Substituting Eq. (22) into Eq. (21), the second integral of the momentum $k_0(r)$ is obtained as

$$\int_{r_{0A}}^{r_{0B}} k_0(r)dr = \frac{\pi}{2\alpha}\left[B + \sqrt{\frac{2\mu R}{\hbar^2}}\right]. \quad (23)$$

The proper quantization rule requires that E_0 in Eq. (21) should be replaced with E_n, therefore we obtain the explicit energy levels of all the bound states as

$$E_n = -V_0 - \frac{2\alpha^2\hbar^2}{\mu}\left[\frac{(\frac{B}{2\alpha}+n)^2 + \frac{2\mu V_0}{\alpha\hbar^2}(\alpha-1)}{2(\frac{B}{2\alpha}+n)}\right]^2, \quad (24)$$

where $B = \alpha[1+\sqrt{(2l+1)^2 - \frac{8\mu V_0}{\hbar^2}}]$. Following the procedure of the formula method,[36] the radial wave function can be written as

$$U(z) = N_{n,l} z^{i\delta/2}(1-z)^{\frac{1}{2}[1+\varepsilon]} \cdot 2F_1(-n, n+i\delta+1+\varepsilon; 1+i\delta, z), \quad (25)$$

where $\delta = \frac{1}{\alpha}\sqrt{\frac{2\mu}{\hbar^2}(E+V_0)}$, $\varepsilon = \sqrt{(1+2l)^2 - \frac{8\mu V_0}{\hbar^2}}$ and $z = e^{-2\alpha r}$ have been used in the calculation. Equations (24) and (25) are the same as Eqs. (45) and (46) obtained by Ikhdair *et al.*[37] The wave function obtained here is not purely imaginary but it consists of two components (real and imaginary parts), which are determined by the shape or nature of the potential under consideration. The square modulus (probability distribution) of the wave function is of physical interest.

Table 1. The bound state energy levels (in units of fm^{-1}) of the GIQYP for various values of n, l and for $\mu = \hbar = 1$.

n	l	E_{nl} $V_0 = 0.5$, $\alpha = 0.001$	E_{nl} $V_0 = 1.0$, $\alpha = 0.001$	E_{nl} $V_0 = 0.5$, $\alpha = 0.01$	E_{nl} $V_0 = 1.0$, $\alpha = 0.01$
0	1	−0.6896035396138025	−2.994004500000000	−0.6774133455125515	−2.940450000000000
1	1	−0.5718076188414074	−1.497004500000000	−0.5619400365376546	−1.470450000000000
2	1	−0.5371277912043335	−1.219784500000000	−0.5281909972598103	−1.198450000000000
3	1	−0.5224100938146438	−1.122760125000000	−0.5141450483150030	−1.103512500000000
0	2	−0.5630508543825075	−1.302201872184403	−0.5533867400061118	−1.279268338661399
1	2	−0.5337240097533026	−1.155363939225181	−0.5249118232916035	−1.135367365171019
2	2	−0.5207493205425896	−1.093938232041968	−0.5125947736790960	−1.075445608618366
3	2	−0.5138959792881225	−1.062548531648839	−0.5063882922242523	−1.045120125270810
0	3	−0.5326018749170197	−1.143685816860237	−0.5238335921350012	−1.123949057454857
1	3	−0.5201906956637058	−1.088310737089324	−0.5120761646075057	−1.069983262515873
2	3	−0.5135787736130005	−1.059418943879603	−0.5061129917794308	−1.042124745183449
3	3	−0.5096463318898519	−1.042468019323530	−0.5028802557635820	−1.026091932802665
0	4	−0.5199025668480347	−1.085664968464422	−0.5118093030597658	−1.067417988764047
1	4	−0.5134143288870058	−1.057929861731356	−0.5059709010589428	−1.040702385388879
2	4	−0.5095438530724833	−1.041548732718895	−0.5028023247194076	−1.025236123010328
3	4	−0.5070530264169021	−1.031076397206152	−0.5010876689076859	−1.015671686891542

The domain in which the screening parameter $\alpha \to 0\,\mathrm{fm}^{-1}$ is called the low screening regime. In this regime the generalized inverse quadratic Yukawa potential model becomes the inverse quadratic Yukawa potential model. For a fixed value of angular momentum quantum l, the energy spectrum increases as the principal quantum number n increases for the weak potential coupling strength, $V_0 = 0.5\,\mathrm{fm}^{-1}$ and $V_0 = 1.0\,\mathrm{fm}^{-1}$. For a fixed value of angular momentum quantum l, the energy spectrum increases as the principal quantum number n increases for a very small screening parameter (i.e., low screening regime) $\alpha = 0.001\,\mathrm{fm}^{-1}$ and $\alpha = 0.01\,\mathrm{fm}^{-1}$. An increase in angular momentum quantum l, leads to an increase in the energy spectrum as the principal quantum number n increases for a very small screening parameter α and for weak potential coupling strength, V_0. For strong potential coupling strength, $V_0 = 5.0\,\mathrm{fm}^{-1}$, solutions are ignored due to the presence of imaginary terms and the energy spectrum is not complex but real.

Figures 1–3 illustrate the dependence of the particle's energy on the potential parameters, which in actual sense determine the state and nature of bounding. There are singularities in Figs. 2 and 3 for $l = 1$ and $l = 2$ but not for $l = 0$ because $E_{nl} \to 0$ when $l = 0$, thereby triggering off the particle towards con-

tinuum state. Accordingly, the spectrum of positive eigenvalues of energy is continuous (and the levels are not degenerate), while the spectrum of negative eigenvalues is discrete (see Ref. [1]). In Figs. 1(a) and 1(b), we plot the energy of the ground $n = 0$ and the first excited $n = 1$ states, respectively, for various values of $l(l = 0, 1, 2)$ as a function of potential screening parameter α. From the plot, we can observe that for $\mu = 1\,\mathrm{fm}^{-1}$ and $V_0 = 1\,\mathrm{fm}^{-1}$, the energy curve is strongly bounded and an increase in rotational quantum number l makes energy become more attractive (i.e., the energy becomes more negative) with increasing α.

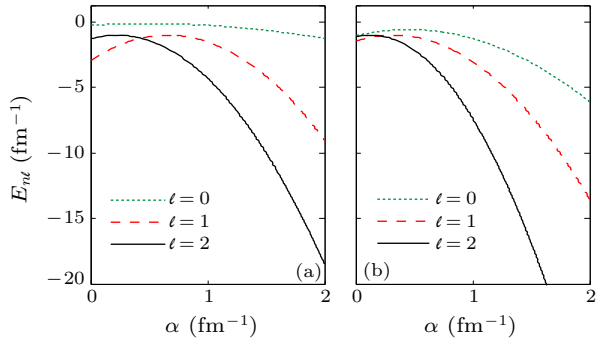

Fig. 1. (a) The behavior of the ground state ($n = 0$) energy level for various $l(l = 0, 1, 2)$ as a function of the potential screening parameter $\alpha(\mathrm{fm}^{-1})$. (b) The behavior of the first excited ($n = 1$) energy state for various $l(l = 0, 1, 2)$ as a function of the potential screening parameter $\alpha(\mathrm{fm}^{-1})$. Here we choose $\mu = 1\,\mathrm{fm}^{-1}$ and $V_0 = 1\,\mathrm{fm}^{-1}$.

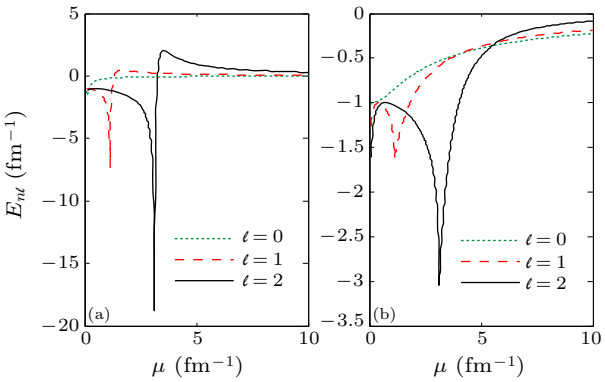

Fig. 2. (a) The behavior of the ground state ($n = 0$) energy level for various $l(l = 0, 1, 2)$ as a function of the particle mass $\mu(\mathrm{fm}^{-1})$. (b) The behavior of the first excited ($n = 1$) energy state for various $l(l = 0, 1, 2)$ as a function of the particle mass $\mu(\mathrm{fm}^{-1})$. We choose $\alpha = 0.1\,\mathrm{fm}^{-1}$ and $V_0 = 1\,\mathrm{fm}^{-1}$.

In Figs. 2(a) and 2(b), we have plotted the energy of the ground $n = 0$ and the first excited $n = 1$ states, respectively, for various values of $l(l = 0, 1, 2)$ as a function of the particle mass μ. From the plot, we observe that for $\alpha = 0.1\,\mathrm{fm}^{-1}$ and $V_0 = 1\,\mathrm{fm}^{-1}$, as the rotational quantum number l increases, the attractive energy speeds up towards the positive energy and later approaches a continuum state. In Figs. 3(a) and 3(b), we have plotted energy levels E_{nl} versus potential depth or coupling strength V_o for $n = 0$ and $n = 1$ states, respectively. For $n = 1$ state, we observe that for $\mu = 1\,\mathrm{fm}^{-1}$ and $\alpha = 0.1\,\mathrm{fm}^{-1}$, the energy decreases and the particle is bounded for $V_o < 4\,\mathrm{fm}^{-1}$ and then becomes less attractive as l increases with an increase in the depth or coupling strength V_o. For $n = 0$, as l increases the particle becomes attractive or the energy is positive (approaching continuum states). However, for $V_o < 4\,\mathrm{fm}^{-1}$, the particle is strongly bounded since the increasing V_o shields the GIQYP field.

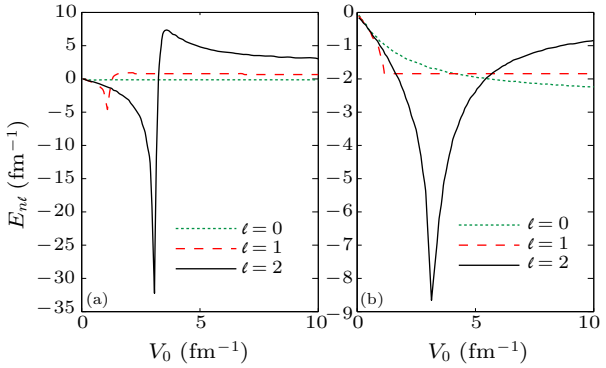

Fig. 3. (a) The behavior of the ground state ($n = 0$) energy level for various $l(l = 0, 1, 2)$ as a function of the potential depth or coupling strength $V_0(\mathrm{fm}^{-1})$. (b) The behavior of the first excited ($n = 1$) energy state for various $l(l = 0, 1, 2)$ as a function of the potential depth or coupling strength $V_0(\mathrm{fm}^{-1})$. We choose $\mu = 1\,\mathrm{fm}^{-1}$ and $\alpha = 0.1\,\mathrm{fm}^{-1}$.

In summary, the bound state solutions to the Schrödinger equation with the generalized inverse quadratic Yukawa potential have been studied within a Pekeris-type approximation. The eigenvalues and eigenfunctions are obtained using proper quantization rule and formula method. The essence of using the formula method is owed to the fact that the proper quantization rule cannot be used to calculate the eigenfunctions. The numerical values of energy eigenvalues are obtained for different states. The ground state energies differ from those of first excited states (see Figs. 1–3). Finally, the results from the table and the plots reveal that the energy eigenvalues, angular momentum numbers, screening parameters, reduced mass, and the potential depth or potential coupling strength determine the nature of bound state of quantum particles. It is also implied that the explored model is suitable for explaining both the bound and continuum states of quantum systems.

We thank the kind referees for the invaluable comments and suggestions that have improved this manuscript.

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