Notes on Canonical Forms of Integrable Vector Nonlinear Schrödinger Systems *

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We present canonical forms of integrable vector nonlinear Schrödinger systems. Mathematically, it is enough to focus on these canonical forms.

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Since the nonlinear Schrödinger (NLS) equation was derived for describing water waves, plasma and nonlinear optics,[1−4] the equation and its relatives (e.g., the Manakov equations) have been playing important roles in nonlinear physics and mathematics and have received much attention (see Ref. [5] and the references therein). A generalized integrable coupled NLS equation was presented in Ref. [6] in the form of

$$iq_{1,t} + q_{1,xx} + q_1(a|q_1|^2 + c|q_2|^2 + bq_1q_2^* + b^*q_1^*q_2) = 0, \tag{1a}$$

$$iq_{2,t} + q_{2,xx} + q_2(a|q_1|^2 + c|q_2|^2 + bq_1q_2^* + b^*q_1^*q_2) = 0, \tag{1b}$$

where i is the imaginary unit, $q_j = q_j(x,t)$, $|q_j| = q_jq_j^*$, $*$ stands for the complex conjugate, $a, c \in \mathbb{R}$, $b \in \mathbb{C}$ and $ac - |b|^2 \neq 0$. As a generalized version of the Manakov system,[7] Eq. (1) has been investigated from many aspects in recent years.[8−11] It is true that choosing different parameters in $\{a, b, c\}$ may result in different dynamical behavior of solutions,[6,8−10] but there are canonical forms of Eq. (1) and mathematically it is enough to consider these canonical forms.

In this work, we present canonical forms of the following vector NLS equation

$$i\boldsymbol{q}_t + \boldsymbol{q}_{xx} + 2\boldsymbol{q}(\boldsymbol{q}^{\mathrm{T}}H\boldsymbol{q}^*) = 0, \tag{2}$$

where $\boldsymbol{q}$ is an Nth-order column vector function $\boldsymbol{q} = (q_1, q_2, \ldots, q_N)^{\mathrm{T}}$ and $H \in \mathbb{C}_{N\times N}$ is a Hermitian matrix, i.e., $H = H^\dagger$. When $N = 2$, Eq. (2) yields Eq. (1) with $H = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$. To our knowledge, Eq. (2) was first explicitly written out by Zakharov and Schulman[12] in 1982 when they investigated possible forms of two-component coupled NLS equations with enough conserved integrals. Before we discuss canonical forms of Eq. (2), let us first recall its Lax integrability. Consider the following generalized Zakharov–Shabat–Ablowitz–Kaup–Newell–Segur (ZS-AKNS) spectral problem[7,13]

$$\Phi_x = M\Phi, \tag{3a}$$

with

$$M = \begin{pmatrix} i\eta E_N & \boldsymbol{q} \\ \boldsymbol{r}^{\mathrm{T}} & -i\eta \end{pmatrix}, \tag{3b}$$

where $\boldsymbol{q}$ and $\boldsymbol{r}$ are Nth-order column vectors $\boldsymbol{q} = (q_1, q_2, \ldots, q_N)^{\mathrm{T}}$, $\boldsymbol{r} = (r_1, r_2, \ldots, r_N)^{\mathrm{T}}$, Φ is an $(N+1)$th-order column vector, E_N stands for an Nth-order identity matrix, and η is a spectral parameter. The compatibility between Eq. (3a) and

$$\Phi_t = W\Phi \tag{4a}$$

with

$$W = \eta^2 \begin{pmatrix} -2iE_N & \mathbf{0}_{N\times 1} \\ \mathbf{0}_{1\times N} & 2i \end{pmatrix} + \eta \begin{pmatrix} \mathbf{0}_{N\times N} & -2\boldsymbol{q} \\ -2\boldsymbol{r}^{\mathrm{T}} & 0 \end{pmatrix} + \begin{pmatrix} -i\boldsymbol{q}\boldsymbol{r}^{\mathrm{T}} & \mathbf{0}_{N\times 1} \\ \mathbf{0}_{1\times N} & i\boldsymbol{q}^{\mathrm{T}}\boldsymbol{r} \end{pmatrix}, \tag{4b}$$

i.e.,

$$M_t - W_x + MW - WM = 0, \tag{5}$$

yields a second-order two-vector system[6]

$$i\boldsymbol{q}_t + \boldsymbol{q}_{xx} - 2\boldsymbol{q}(\boldsymbol{q}^{\mathrm{T}}\boldsymbol{r}) = 0, \tag{6a}$$

$$i\boldsymbol{r}_t - \boldsymbol{r}_{xx} + 2\boldsymbol{r}(\boldsymbol{q}^{\mathrm{T}}\boldsymbol{r}) = 0. \tag{6b}$$

This admits a reduction[6,12]

$$\boldsymbol{r}(x,t) = -H\boldsymbol{q}^*(x,t), \tag{7}$$

which yields Eq. (2) and nonlocal reduction

$$\boldsymbol{r}(x,t) = H\boldsymbol{q}^*(-x,t), \tag{8}$$

which was first introduced in Ref. [14] for the scalar case, yielding

$$i\boldsymbol{q}_t(x,t) + \boldsymbol{q}_{xx}(x,t) + 2\boldsymbol{q}(x,t)(\boldsymbol{q}^{\mathrm{T}}(x,t)H\boldsymbol{q}^*(-x,t)) = 0, \tag{9}$$

with $H = H^\dagger$ being an Nth-order constant Hermitian matrix.

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For the normalization and canonical forms of the vector NLS system (2) and (9), we have the following results.

Theorem 1: The vector NLS system (2) has the canonical forms of

$$i\boldsymbol{q}_t + \boldsymbol{q}_{xx} + 2\boldsymbol{q}(\sum_{j=1}^{N}\delta_j|q_j|^2) = 0, \qquad (10)$$

and the nonlocal vector NLS system (9) has the canonical forms of

$$i\boldsymbol{q}_t(x,t) + \boldsymbol{q}_{xx}(x,t) + 2\boldsymbol{q}(x,t)(\sum_{j=1}^{N}\delta_i q_j(x,t)q_j^*(-x,t)) = 0, \qquad (11)$$

where $\delta_j = \pm 1$ and $\delta_j \leq \delta_{j+1}$.

Proof: Based on the complex inertial theorem,[15] the Hermitian matrix H holds the diagonalization

$$H = U^{\rm T} \Lambda U^*, \qquad (12)$$

where $\Lambda = {\rm Diag}\{\lambda_1, \ldots, \lambda_N\}$, and U is a unitary matrix. Since H is a Hermitian matrix, all its eigenvalues $\{\lambda_i\}$ are real and can be ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Introducing a new matrix $V = {\rm Diag}\{|\lambda_1|^{1/2}, \ldots, |\lambda_N|^{1/2}\}U$, such that H is written as

$$H = V^{\rm T} E_{[S]} V^*, \qquad (13)$$

where the $E_{[S]}$ is denoted by

$$E_{[S]} = \begin{pmatrix} E_S & \boldsymbol{0}_{S\times(N-S)} \\ \boldsymbol{0}_{(N-S)\times S} & -E_{N-S} \end{pmatrix}, \qquad (14)$$

and S is the positive inertial index for H. With $\boldsymbol{u} = V\boldsymbol{q}$, Eq. (6) is normalized as

$$i\boldsymbol{u}_t + \boldsymbol{u}_{xx} + 2\boldsymbol{u}(\boldsymbol{u}^{\rm T} E_{[S]} \boldsymbol{u}^*) = 0, \qquad (15)$$

which is nothing but Eq. (10) with $\boldsymbol{q}$ in place of $\boldsymbol{u}$. In a similar way, the nonlocal system (9) is normalized as Eq. (11).

We note that once Eq. (15) is solved, left-multiplied by an arbitrary Hermitian matrix $J = J^\dagger$, then $\boldsymbol{q} = J\boldsymbol{u}$ can recover Eq. (2) with $H = J^{\rm T} E_{[S]} J^*$. The same relation holds for Eqs. (9) and (11).

According to the Theorem, when $N = 2$, Eq. (2) has the three canonical forms as follows:

$$iq_{j,t} + q_{j,xx} - 2q_j(|q_1|^2 + |q_2|^2) = 0, \ j = 1, 2, \qquad (16)$$
$$iq_{j,t} + q_{j,xx} - 2q_j(|q_1|^2 - |q_2|^2) = 0, \ j = 1, 2, \qquad (17)$$
$$iq_{j,t} + q_{j,xx} + 2q_j(|q_1|^2 + |q_2|^2) = 0, \ j = 1, 2. \qquad (18)$$

The three canonical forms of Eq. (9) are

$$iq_{j,t}(x,t) + q_{j,xx}(x,t) - 2q_j(x,t)(q_1(x,t)q_1^*(-x,t) + q_2(x,t)q_2(-x,t)) = 0, \ j = 1, 2, \qquad (19)$$
$$iq_{j,t}(x,t) + q_{j,xx}(x,t) - 2q_j(x,t)(q_1(x,t)q_1^*(-x,t) - q_2(x,t)q_2(-x,t)) = 0, \ j = 1, 2, \qquad (20)$$
$$iq_{j,t}(x,t) + q_{j,xx}(x,t) + 2q_j(x,t)(q_1(x,t)q_1^*(-x,t) + q_2(x,t)q_2(-x,t)) = 0, \ j = 1, 2. \qquad (21)$$

Equations (16)–(18) compose a full list of integrable normalized two-component NLS systems that are related to the spectral problem (3) in which $\boldsymbol{q}$ and $\boldsymbol{r}$ are vectors.

In the following let us look for more results on integrable two-component NLS systems. In Ref. [16] a searching approach was employed to find integrable two-component NLS systems of the form

$$iq_{1,t} + c_1 q_{1,xx} + q_1(\alpha|q_1|^2 + \beta|q_2|^2) + \delta q_1^* q_2^2 = 0,$$
$$iq_{2,t} + c_2 q_{2,xx} + q_2(\beta|q_1|^2 + \gamma|q_2|^2) + \delta q_1^2 q_2^* = 0,$$

which is required to have enough conserved quantities, where coefficients $c_i, \alpha, \beta, \gamma, \delta \in \mathbb{C}$. The exhausted integrable list (up to simple transformations) found by Ref. [16] includes Eqs. (16)–(18) and four more systems $\{K_1^+, K_2^+\}$, $\{K_1^-, K_2^-\}$, $\{H_1^+, H_2^+\}$ and $\{H_1^-, H_2^-\}$, where

$$K_1^\pm: iq_{1,t} + q_{1,xx} \pm [q_1(|q_1|^2 + 2|q_2|^2) + q_1^* q_2^2] = 0, \qquad (22a)$$
$$K_2^\pm: iq_{2,t} + q_{2,xx} \pm [q_2(2|q_1|^2 + |q_2|^2) + q_1^2 q_2^*] = 0, \qquad (22b)$$
$$H_1^\pm: iq_{1,t} + q_{1,xx} \pm [q_1(|q_1|^2 + 2|q_2|^2) - q_1^* q_2^2] = 0, \qquad (23a)$$
$$H_2^\pm: iq_{2,t} + q_{2,xx} \pm [q_2(2|q_1|^2 + |q_2|^2) - q_1^2 q_2^*] = 0. \qquad (23b)$$

For these systems, we have the following comments. Firstly, these systems are related to two matrix-type spectral problems[13] that are different from Eq. (3). For example, taking $q_3 = q_1$ and $q_4 = q_2$ in (3.20) and (3.26) in Ref. [13] one can immediately obtain K_j^+ and H_j^+. Secondly, $H_j^\pm$ can be derived from $K_j^\pm$ by simply replacing q_2 with iq_2. Therefore one only needs to consider $K_j^\pm$. Thirdly, systems $K_j^\pm$ are related to a matrix NLS system from a viewpoint of a group. Consider the matrix NLS system[5]

$$i\boldsymbol{Q}_t + \boldsymbol{Q}_{xx} \pm \boldsymbol{Q}\boldsymbol{Q}^\dagger\boldsymbol{Q} = \boldsymbol{0}, \qquad (24)$$

where $\boldsymbol{Q}$ belongs to the group $G = \{\boldsymbol{A} | , \boldsymbol{A} \in \mathbb{C}_{2\times 2}(x,t), |\boldsymbol{A}| \neq 0\}$. Also consider the following subgroups $G^\pm = \{\begin{pmatrix} a & b \\ \pm b & a \end{pmatrix} \in G\}$. Noting that for all $\boldsymbol{Q} \in G^\pm$, $\boldsymbol{Q}^\dagger$ belongs to the same subgroup, we can take $\boldsymbol{Q} = \begin{pmatrix} q_1 & q_2 \\ \pm q_2 & q_1 \end{pmatrix}$, and the matrix NLS system (24) yields $K_j^{\pm}$ and $H_j^{\pm}$, respectively. Finally, we point out that $K_j^{\pm}$ is actually trivial. In fact, taking K_j^{+} as an example, introducing

$$q_1 = u + v, \quad q_2 = u - v, \tag{25}$$

we can rewrite K_j^{+} into the form

$$K_1^{+} : \{iu_t + u_{xx} + 4u|u|^2\} + \{iv_t + v_{xx} + 4v|v|^2\} = 0,$$
$$K_2^{+} : \{iu_t + u_{xx} + 4u|u|^2\} - \{iv_t + v_{xx} + 4v|v|^2\} = 0,$$

which means that the K_j^{+} system is trivially determined by the scalar NLS equation

$$iq_t + q_{xx} + 4q|q|^2 = 0. \tag{26}$$

In summary, we have presented canonical forms of the integrable vector NLS systems (2) and (9) in theorem 1. As for the two-component case, the general coupled NLS system (1) has been over-investigated. It is worth pointing out that mathematically we only need to focus on these canonical forms. For the vector NLS Eq. (2), in the two-component case there are three canonical forms: Eqs. (16)–(18). In fact, more or less, these canonical forms have been investigated from different aspects, e.g., see Refs. [17–20]. Although the two-component systems (22) and (23) are related to the matrix NLS Eq. (24) by special reductions and they appeared in Ref. [16] as a searching result, they can be trivially determined by the scalar NLS Eq. (26). In conclusion, in the two-component case, the canonical forms for investigation are Eqs. (16)–(18), and for the nonlocal case they are Eqs. (19)–(21).

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