Alice–Bob Peakon Systems *

Sen-Yue Lou(楼森岳)[1,2], Zhi-Jun Qiao(乔志军)[3**]

[1]Center for Nonlinear Science and Department of Physics, Ningbo University, Ningbo 315211

[2]Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai 200062

[3]School of Mathematical and Statistical Science, University of Texas, TX 78539, USA

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We study the Alice–Bob peakon system generated from an integrable peakon system using the strategy of the so-called Alice–Bob non-local KdV approach [Scientific Reports 7 (2017) 869]. Nonlocal integrable peakon equations are obtained and shown to have peakon solutions.

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Recently, nonlocal integrable systems have attracted much attention since the pioneer work about nonlocal nonlinear Schrödinger equation.[1] Lou and Huang applied the nonlocal approach to the KdV equation and proposed the so-called Alice–Bob (AB) physics to generate coherent solutions of nonlocal KdV systems.[2] Before discussing AB peakon systems, let us first recall the basic procedure for regular AB integrable systems.

In the study of classical physical theory, most feasible models are locally established around a single space-time point, namely, $\{x, t\}$. To investigate some related physical phenomena in two or more places, one has to consider some kinds of new models. In Refs. [2,3], the authors proposed Alice–Bob (AB) models to study two-place physical problems. Alice–Bob physics will make sense if the physics is related to two entangled events occurring in two places $\{x, t\}$ and $\{x', t'\}$. The event at $\{x, t\}$ is called the Alice event (AE) (denoted by $A(x, t)$) and the event at $\{x', t'\}$ is called the Bob event (BE) (denoted by $B(x', t')$). The events AE and BE will be correlated/entangled if AE happens, BE can be determined immediately by the correlation condition

$$B(x', t') = f(A) = A^f = \hat{f}A, \tag{1}$$

where $\hat{f}$ represents a suitable operator, which may be different at different events. In general, $\{x', t'\}$ is not a neighbor to $\{x, t\}$. Thus the intrinsic two-place models, the Alice–Bob systems (ABSs), are nonlocal. Some special types of two-place nonlocal models have been proposed. For example, the nonlocal nonlinear Schrödinger (NLS) equation

$$iA_t + A_{xx} \pm A^2 B = 0, \quad B = \hat{f}A = \hat{P}\hat{C}A = A^*(-x, t)$$

was firstly proposed by Ablowitz and Musslimani.[1] This type of NLS system has strong relationships to the significant PT symmetric Schrödinger equations.[4] The operators $\hat{P}$ and $\hat{C}$ are the usual parity and charge conjugation, respectively. Afterwards, other nonlocal systems, including the coupled nonlocal NLS systems,[5] the nonlocal modified KdV systems,[6,7] the discrete nonlocal NLS system,[8] and the nonlocal Davey–Stewartson systems,[9–11] were investigated as well.

In this work, we utilize the Alice–Bob approach to study integrable peakon equations. For our convenience, the peakon system based on the Alice–Bob (AB) approach is called the Alice–Bob peakon (ABP) system. We will take the integrable peakon systems studied in Ref. [12] as the ABP examples.

Let us consider the following model

$$\begin{aligned} m_t &= (mH)_x + mH - \frac{1}{2}m(A - A_x)(\bar{A} + \bar{A}_x), \\ \bar{A} &= A(-x + x_0, \ -t + t_0), \qquad (2) \\ m &= A - A_{xx}, \qquad (3) \end{aligned}$$

where H is an arbitrary shifted parity ($\hat{P}_s$) and delayed time reversal ($\hat{T}_d$) invariant functional

$$\hat{P}_s\hat{T}_d H = \bar{H} = H, \tag{4}$$

and the definitions of $\hat{P}_s$ and $\hat{T}_d$ read $\hat{P}_s x = -x + x_0$ and $\hat{T}_d t = -t + t_0$. Actually, Eq. (2) is generated through the first equation of the two-component system (7) proposed in Ref. [12] via $u = A$ and $v = \bar{A}$.

Through a lengthy computation, one may have a Lax pair for Eq. (2),

$$\phi_x = U\phi, \ U = \begin{pmatrix} -1 & \lambda m \\ -\lambda\bar{m} & 1 \end{pmatrix}, \ \bar{m} = \bar{A} - \bar{A}_{xx}, \tag{5}$$

$$\phi_t = V\phi, \quad V = \frac{1}{2}\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \tag{6}$$

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**Corresponding author. Email: zhijun.qiao@utrgv.edu

where $V_{11} = \frac{1}{2}(A_x - A)(\bar{A} + \bar{A}_x) - \lambda^{-2}$, $V_{12} = \lambda^{-1}(A - A_x) + \lambda mH$, $V_{21} = -\lambda^{-1}(\bar{A} + \bar{A}_x) - \lambda\bar{m}H$, and $V_{22} = \lambda^{-2} + \frac{1}{2}(A - A_x)(\bar{A} + \bar{A}_x)$.

Following the typical procedure starting from Lax pair, we may obtain the following conservation laws for Eq. (2),

$$\rho_{\rm jt} = J_{\rm jx},\ j = 0, 1, 2, \ldots, \infty, \tag{7}$$
$$\rho_j = m\omega_j,\ j = 0, 1, 2, \ldots, \tag{8}$$
$$J_j = (A - A_x)\omega_{j-2} + H\rho_j,\ j = 2, 3, \ldots, \tag{9}$$
$$J_0 = H\rho_0,\ J_1 = H\rho_1 - \frac{1}{2}(A - A_x)(\bar{A} + \bar{A}_x), \tag{10}$$
$$\omega_{j+1} = \frac{1}{m\omega_0}\Big(\omega_j - \omega_{\rm jx} - \frac{m}{2}\sum_{k=1}^{j}\omega_k\omega_{j+1-k}\Big),\quad j = 1, 2, \ldots, \tag{11}$$
$$\omega_0 = \sqrt{-\bar{m}m^{-1}},\ \omega_1 = \frac{m\bar{m}_x - \bar{m}m_x - 2m\bar{m}}{2m^2\bar{m}}. \tag{12}$$

It is noticed that the conserved densities $\rho_j = m\omega_j, j = 0, 1, \ldots$ are not explicitly dependent on H.

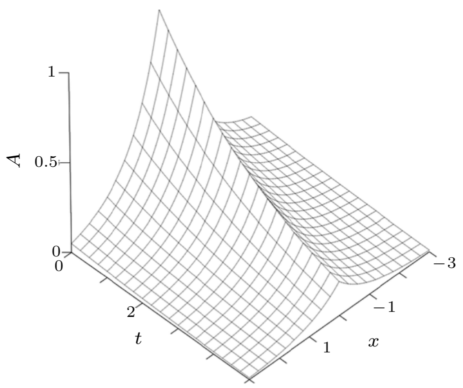

Fig. 1. The fast decayed non-traveling peakon solution (13) for the first special ABP system (13) with the peakon parameters $x_0 = t_0 = 0$ and $c = 1$.

Let us now solve concrete peakon solutions to Eq. (2) for some special functions H listed in the following examples.

Example 1: Take $H = 0$ to send Eq. (2) to the following system

$$\begin{cases} m_t = -\frac{1}{2}m(A - A_x)(\bar{A} + \bar{A}_x), \\ m = A - A_{xx},\ \bar{A} \equiv A(-x + x_0, -t + t_0). \end{cases}$$

This system has only one-peakon solutions

$$A = c\exp\Big[-\frac{1}{3}c^2\Big(t - \frac{t_0}{2}\Big) - \Big|x - \frac{x_0}{2}\Big|\Big], \tag{13}$$

which are non-traveling solitary waves with a fast decayed standing peak, as shown in Fig. 1.

No multi-peakon solution is found for this special example though it is integrable with the Lax pair and infinitely many conservation laws.

Example 2: Selecting $H = \frac{1}{2}(A\bar{A} - A_x\bar{A}_x)$ leads Eq. (2) to

$$\begin{cases} m_t = \frac{1}{2}[m(A\bar{A} - A_x\bar{A}_x)]_x - \frac{1}{2}m(A - A_x)(\bar{A} + \bar{A}_x), \\ m = A - A_{xx},\ \bar{A} \equiv A(-x + x_0,\ -t + t_0). \end{cases} \tag{14}$$

This equation possesses the following one-peakon solutions

$$A = c\exp\Big[-\Big|\Big(x - \frac{x_0}{2}\Big) + \frac{1}{3}c^2\Big(t - \frac{t_0}{2}\Big)\Big|\Big]. \tag{15}$$

as well as the N-peakon dynamical system

$$A = \sum_{j=1}^{N} p_j \exp\Big(-\Big|x - \frac{x_0}{2} - q_j\Big|\Big), \tag{16}$$
$$p_{\rm jt} = \frac{1}{2}p_j\sum_{i,k=1}^{N} p_i\bar{p}_k[\mathrm{sgn}(q_j - q_k) - \mathrm{sgn}(q_j - q_i)]e^{-|q_j - q_k| - |q_j - q_i|}, \tag{17}$$
$$q_{\rm jt} = \frac{1}{6}p_j\bar{p}_j - \frac{1}{2}\sum_{i,k=1}^{N} p_i\bar{p}_k[1 - \mathrm{sgn}(q_j - q_k) \cdot \mathrm{sgn}(q_j - q_i)]e^{-|q_j - q_k| - |q_j - q_i|}. \tag{18}$$

Though the integrability problem of the above N-peakon system is open, we obtain the following explicit 2-peakon solutions

$$p_{1t} = \frac{1}{2}p_1(p_1\bar{p}_2 - p_2\bar{p}_1)\mathrm{sgn}(q_1 - q_2)e^{-|q_1 - q_2|}), \tag{19}$$
$$p_{2t} = \frac{1}{2}p_2(p_2\bar{p}_1 - p_1\bar{p}_2)\mathrm{sgn}(q_2 - q_1)e^{-|q_2 - q_1|}), \tag{20}$$
$$q_{1t} = -\frac{1}{3}p_1\bar{p}_1 - \frac{1}{2}(p_1\bar{p}_2 + p_2\bar{p}_1)e^{-|q_1 - q_2|}, \tag{21}$$
$$q_{2t} = -\frac{1}{3}p_2\bar{p}_2 - \frac{1}{2}(p_1\bar{p}_2 + p_2\bar{p}_1)e^{-|q_2 - q_1|}. \tag{22}$$

Therefore,

$$q_1 = \frac{3p_1p_2\mathrm{sgn}(2t - t_0)}{|p_1^2 - p_2^2|}[e^{-\frac{1}{6}|(p_1^2 - p_2^2)(2t - t_0)|} - 1] - \frac{p_1^2}{6}(2t - t_0),$$
$$q_2 = \frac{3p_1p_2\mathrm{sgn}(2t - t_0)}{|p_1^2 - p_2^2|}[e^{-\frac{1}{6}|(p_1^2 - p_2^2)(2t - t_0)|} - 1] - \frac{p_2^2}{6}(2t - t_0), \tag{23}$$

where p_1 and p_2 are two arbitrary constants.

Figure 2 exhibits the single steady traveling peakon solution expressed by Eq. (15), and Fig. 3 shows the interactional behavior for the two-peakon system given by Eq. (16) with $N = 2$ and Eq. (23).

Example 3: Choose $H = \frac{1}{2}(A\bar{A}_x - A_x\bar{A})$ to cast Eq. (2) to the following system

$$\begin{cases} m_t = \frac{1}{2}[m(A\bar{A}_x - A_x\bar{A})]_x - \frac{1}{2}m[A\bar{A} - A_x\bar{A}_x], \\ m = A - A_{xx},\ \bar{A} \equiv A(-x + x_0,\ -t + t_0), \end{cases} \tag{24}$$

which has fast decayed one-peakon solutions as follows:

$$A = c\exp\Big[-\frac{1}{3}c^2\Big(t - \frac{t_0}{2}\Big) - \Big|x - \frac{x_0}{2}\Big|\Big]. \tag{25}$$

This single peakon possesses the same form as the one in example 1.

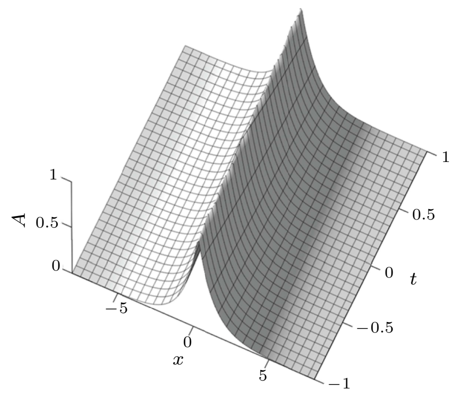

Fig. 2. The single steady traveling peakon solution (15) for the second special ABP system (14) with the peakon parameters $x_0 = t_0 = 0$ and $c = 1$.

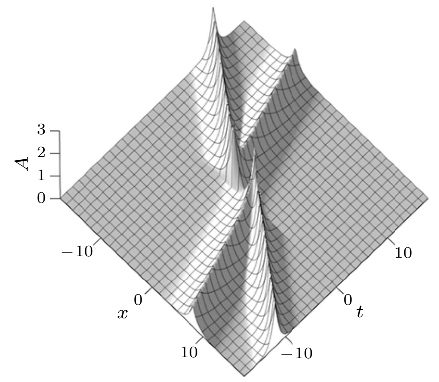

Fig. 3. The two peakon interactional solution (16) for the second special ABP system (14). The corresponding parameters are chosen as $N = 2$, $x_0 = t_0 = 0$, $p_1 = 1$ and $p_2 = 2$.

Similar to example 1 we have not yet found two-peakon solutions, though the model is also integrable.

Example 4: Let $H = \frac{1}{2}(A - A_x)(\bar{A} + \bar{A}_x)$. Then we obtain the following system

$$\begin{cases} m_t = \frac{1}{2}[m(A - A_x)(\bar{A} + \bar{A}_x)]_x, \\ m = A - A_{xx}, \ \bar{A} \equiv A(-x + x_0, \ -t + t_0), \end{cases} \tag{26}$$

which admits the following one-peakon solutions

$$A = c \exp\Big[- \Big| \Big(x - \frac{x_0}{2}\Big) + \frac{1}{3}c^2\Big(t - \frac{t_0}{2}\Big)\Big|\Big]. \tag{27}$$

Furthermore, we can obtain the N-peakon dynamical system

$$A = \sum_{j=1}^{N} p_j \exp\Big(- \Big| x - \frac{x_0}{2} - q_j \Big| \Big), \tag{28}$$

$$p_{jt} = 0, \tag{29}$$

$$q_{jt} = \frac{1}{6} p_j \bar{p}_j - \frac{1}{2} \sum_{i,k=1}^{N} p_i \bar{p}_k [\mathrm{sgn}(q_j - q_k) - 1] \cdot [\mathrm{sgn}(q_j - q_i)] + 1] e^{-|q_j - q_k| - |q_j - q_i|}. \tag{30}$$

In particular, two-peakon solutions take on the following form

$$q_1 = \frac{1}{6} p_1^2 (2t - t_0) + \frac{3 p_1 p_2 \mathrm{sgn}(2t - t_0)}{|p_1^2 - p_2^2|} \cdot (e^{-\frac{1}{6}|(p_1^2 - p_2^2)(2t - t_0)|} - 1), \tag{31}$$

$$q_2 = \frac{1}{6} p_2^2 (2t - t_0) + \frac{3 p_1 p_2 \mathrm{sgn}(2t - t_0)}{|p_1^2 - p_2^2|} \cdot (e^{-\frac{1}{6}|(p_1^2 - p_2^2)(2t - t_0)|} - 1), \tag{32}$$

where p_1 and p_2 are two arbitrary constants. In this case, both the single peakon and two-peakon forms are the same as those in example 2.

In this study, we give a quite general integrable Alice–Bob peakon system (2) with an arbitrary $\hat{P}_s\hat{T}_d$ invariant functional. The existence of multi-peakon solutions are discussed for some special cases by selecting different $\hat{P}_s\hat{T}_d$ invariant functionals. The Lax pair of the ABP system (2) and infinitely many conservation laws are found. Starting from the Lax pair and the conservation laws, one may readily set up other integrable properties such as the infinitely many symmetries, bi-Hamiltonian structures, recursion operator, and inverse scattering transformation. However, we do not discuss those topics here, instead, we focus on peakon solutions for the ABP system. It already reveals from examples 1 and 3 that for the ABP systems if the single peakon is non-travelling (standing) and fast decayed, then there may be no two-peakon interactional solutions. If the single peakon is non-standing (travelling) and not decayed like examples 2 and 4, then there may be multi-peakon solutions. Another amazing fact is that for different ABP systems like examples 2 and 4, their multi-peakon solutions may be completely the same.

There are some intrinsic differences between usual peakon systems and ABP systems. The main difference is that usual peakon systems are local while the ABP systems are nonlocal. The nonlocal property is introduced because the model includes two far-away correlated events, namely, AE and BE. Due to the intrusion of two far-away events in one model, the invariant property in both space and time translations is broken. This kind of symmetry-breaking property probably destroys the existence of multiple decayed standing peakons. In general, the integrability problem of the multi-AB-peakon system remains open. However, an AB-peakon equation may have a classical (smooth) soliton solution if one considers a non-zero boundary condition at both infinities, which we will attend to elsewhere.

The study of regular peakon systems is one of the hot topics in mathematical physics and many different peakon systems were found in the literature. For every peakon system, there may exist different versions of integrable ABP systems. Here we just list some of them we developed, but leave the details for our near future investigations.

Alice–Bob Cammasa–Holm (ABCH): Based on the well-known Camassa–Holm (CH) equation,[13] one of the integrable ABCH systems can be written as

$$(1-\partial_x^2)A_t = (A+B)(3A-A_{xx})_x - 2(A+B)_x A_{xx} + H,$$

where $B = \hat{P}_s\hat{T}_d A$, and H is an arbitrary $\hat{P}_s\hat{T}_d$ invariant functional.

Alice–Bob Degasperis–Procesi (ABDP): Similar to the ABCH, adopting the well-known DP equation[14] yields the integrable ABDP system in the following form

$$(1-\partial_x^2)A_t = (A+B)(4A-A_{xx})_x - 3(A+B)_x A_{xx} + H,$$

with an arbitrary $\hat{P}_s\hat{T}_d$ invariant functional H and $B = \hat{P}_s\hat{T}_d A$.

Alice–Bob b-family (ABbf): Similar to the ABCH and ABDP and considering the b-family equation,[15] one may generate the ABbf system in the following form

$$\begin{aligned}(1-\partial_x^2)A_t =& (A+B)((b+1)A-A_{xx})_x \\ &- b(A+B)_x A_{xx} + H,\end{aligned}$$

with an arbitrary $\hat{P}_s\hat{T}_d$ invariant functional H and $B = \hat{P}_s\hat{T}_d A$.

Alice–Bob Novikov (ABN): For the Novikov equation,[16] there exists the following ABN system

$$(1-\partial_x^2)A_t = (A+B)^2(A_{xx}-4A)_x + \frac{3}{2}(A+B)_x^2 A_{xx} + H,$$

with $\hat{P}_s\hat{T}_d$ invariant functional H and $B = \hat{P}_s\hat{T}_d A$.

Alice–Bob FORQ (ABFORQ): For the FORQ system,[17–19] we have an integrable ABFORQ extension as follows:

$$(1-\partial_x^2)A_t = \{[(A+B)^2 - (A+B)_x^2](A_{xx}-A)\}_x + H,$$

where H is $\hat{P}_s\hat{T}_d$ invariant and $B = \hat{P}_s\hat{T}_d A$.

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