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Ground State of Fermions in a 1D Trap with δ Function Interaction *

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The ground state of fermions in a 1D trap with δ function interaction is studied mathematically with group theory ideas.

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Marvelous developments in laser cooling and trapping in the last 20 years have created the new field of cold atom research. One exciting subfield[1] is the physics of trapped one-dimensional atoms. A useful model for such systems is the Hamiltonian:

$$H = \sum_{i=1}^{N} \Big[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + V(x_i)\Big] + g\sum_{i>j}\delta(x_i - x_j). \quad (1)$$

We assume the trapping potential $V(x)$ to approach ∞ when $x \to -\infty$ and when $x \to +\infty$. Throughout this paper we consider anharmonic traps as well as harmonic ones.

This Hamiltonian is invariant under the permutation group S_N, and contains only space coordinates $x_1, x_2, \ldots, x_N$. Its eigenfunctions are classified by irreducible representations (IR) of S_N. We designate these IRs by their Young graph Y and adopt a simplified notation. For example, we write $\{3, 2, 1, 1\}$ for the graph

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indicating the number of squares in each column. This is *different* from the standard notation.

By a well known theorem, for fermions with spin 1/2 one first forms spin wavefunctions belonging to the IR Y' (where Y' contains at most 2 rows), and then makes dot product of such spin wavefunctions (and partners under S_N) with corresponding space wavefunctions belonging to IR Y, to form $(2J + 1)$ totally antisymmetric wavefunctions Φ with a total spin equal to one half of the difference of the lengths of the two columns of Y. Wavefunctions formed in such a manner span the complete Hilbert space of the fermion system.

CASE $g = 0$

We now concentrate on the space wavefunctions. Denote the eigenfunctions for the single particle Hamiltonian by $u_n(x)$, $n = 0, 1, 2, \ldots$ with energies $\varepsilon_0 < \varepsilon_1 < \varepsilon_2, \ldots$. When $g = 0$, all eigenfunctions of the full Hamiltonian (1) can be expressed as products of the u's.

This problem has been discussed by Girardeau and Minguzzi.[2] The general problem for $g \neq 0$ is difficult. In an important recent paper, Guan *et al.* found[3] the limit of the ground state energy as $g \to +\infty$, for each symmetry Y. In this paper we examine the nature of the wavefunction for each symmetry Y, and how it approaches a limit as $g \to +\infty$, by converting the problem to an equivalent problem B which *does not have symmetry requirements.*

To find the eigenfunctions belonging to a specific symmetry Y when $g = 0$, we apply the Young operation QP for that Y, on a product wavefunction of N single particles of the form

$$u_a(x_1)u_b(x_2)\cdots. \quad (2)$$

The operator P is a symmetrizer, and Q an antisymmetrizer. We illustrate this process with the case of $N = 6$, for symmetry $Y = \{3, 3\}$. Starting from the space eigenfunction

$$[u_0(x_1)u_1(x_2)u_2(x_3)][u_0(x_4)u_1(x_5)u_2(x_6)], \quad (3)$$

we obtain

$$\begin{aligned}\psi_Y =& 8\det[u_0(x_1)u_1(x_2)u_2(x_3)] \\ &\times \det[u_0(x_4)u_1(x_5)u_2(x_6)], \end{aligned} \quad (4)$$

where we introduce the notation

$$\det[u_0(x_1)u_1(x_2)u_2(x_3)] = \begin{vmatrix} u_0(x_1) & u_1(x_1) & u_2(x_1) \\ u_0(x_2) & u_1(x_2) & u_2(x_2) \\ u_0(x_3) & u_1(x_3) & u_2(x_3) \end{vmatrix}.$$

This ψ is an eigenstate of the Hamiltonian (1) with eigenvalue $(\varepsilon_0 + \varepsilon_1 + \varepsilon_2) + (\varepsilon_0 + \varepsilon_1 + \varepsilon_2)$, belonging to the symmetry $\{3, 3\}$.

If we had started from another product like (3) where between the 6 indices for the u's there are 3

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identical ones, then the resultant ψ is zero because, for example,

$$\det[u_a(x_1)u_a(x_2)u_6(x_3)] = 0.$$

Thus ε_0 can occur at most twice, also ε_1 can occur at most twice, and we conclude that (4) is the *ground state* for Hamiltonian (1).

Wavefunction (4) vanishes on any one of the following 6 planes:

$$x_1 = x_2, \quad x_2 = x_3, \quad x_3 = x_1,$$
$$x_4 = x_5, \quad x_5 = x_6, \text{ and } x_6 = x_4. \qquad (5)$$

Now consider the open region R_Y defined by

$$R_Y : x_1 < x_2 < x_3, \text{ and } x_4 < x_5 < x_6. \qquad (6)$$

Figure 1 shows that the region $x_1 < x_2 < x_3$ is bounded by the two planes $x_1 = x_2$ and $x_2 = x_3$. Thus R_Y is bounded by the 4 planes

$$x_1 = x_2, \ x_2 = x_3, \ x_4 = x_5, \text{ and } x_5 = x_6. \qquad (7)$$

It is important to notice that ψ_Y vanishes on the boundary planes (7) of R_Y. Figure 1 also shows that altogether there are $6 \times 6 = 36$ regions like R_Y defined by (6).

AN EQUIVALENT PROBLEM

We state and prove for space wavefunctions:

Theorem 1: For any value of g, consider two different eigenvalue problems:

(A) Hamiltonian H with symmetry $Y = \{3, 3\}$ in full ∞^6 space, and

(B) Hamiltonian H in region R_Y with the boundary condition that the wavefunction vanishes on its surface (7) [Notice this boundary condition is Y dependent].

The eigenvalues of the two problems are identical, and the corresponding unnormalized wavefunctions are proportional in region R_Y.

Proof: Starting from a solution of problem B, by successive reflections on the four boundary planes (7), the space eigenfuction can be continued into full ∞^6 space. Also, starting from a solution ψ of problem A, the wavefunction QPψ vanishes on the planes (7). Thus if it is nonvanishing, it is an eigenfunction for problem B. QED

[Notice this theorem is the generalization of the simple theorem for two particles: Any antisymmetric wavefunctions in full ∞^2 space can be continued from a wavefunction in half space that vanishes on the boundary $x_1 = x_2$.]

Theorem 2: For any value of g, the ground state wavefunction for problem B has no zeros in the interior of R_Y, and is not degenerate.

Proof: This is a special case of the general theorem that the ground state wavefunction has no zeros in the interior, if we do not impose symmetry conditions. QED

It follows from these two theorems that for symmetry $\{3, 3\}$, the ground state of Hamiltonian (1) is not degenerate.

CASE $g \neq 0$

For $g \neq 0$, we take advantage of Theorem 1 and study eigenvalue problem B instead of the original eigenvalue problem. The advantage of B is that no symmetry condition is imposed. Starting from wavefunction (4) at $g = 0$, we follow ψ_Y as g changes. According to Theorem 2, it is everywhere ≥ 0 in R_Y. It vanishes on the 4 boundary planes (7) of R_Y. Continuing them into the full ∞^6 space gives an eigenfunction of problem A with $Y = \{3, 3\}$.

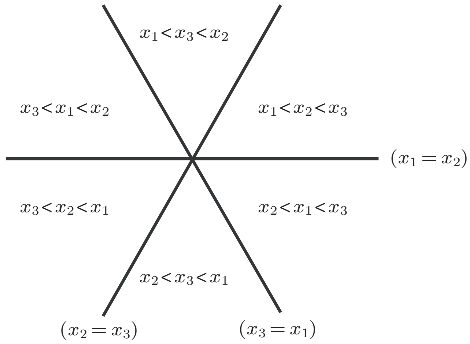

Fig. 1. The 3 planes $x_1 = x_2$, $x_2 = x_3$ and $x_3 = x_1$ divide ∞^3 space into 6 regions, as indicated.

There are altogether 15 delta function interactions in (1). Four of them reside on boundary (7) of R_Y. Figure 1 shows that two of them, $\delta(x_1 - x_3)$ and $\delta(x_4 - x_6)$ are entirely outside of R_Y. The remaining nine reside on nine planes:

$$x_i - x_j = 0, \quad i = 1, 2 \text{ or } 3, \quad j = 4, 5, \text{ or } 6, \qquad (8)$$

each of which partly lies inside region R_Y. Inside R_Y, when g increases, V *shaped cusps* are formed on wavefunction ψ_Y at these nine planes, depressing its value on the plane. But by Theorem 2, ψ_Y remains > 0 in the interior of region R_Y. When $g \to +\infty$, the cusps become infinitely deep, i.e., $\psi_Y \to 0$ on the nine planes inside R_Y. We denote this limiting wavefunction by $\psi_{Y\infty}$. It vanishes only on the nine planes (8) and on the boundary of R_Y [the four planes (7)].

Now the nine planes (8) divide R_Y into $\dfrac{6!}{3!3!} = 20$ subregions, such as

$$x_1 < x_4 < x_2 < x_5 < x_3 < x_6$$
$$\text{or } \ x_4 < x_5 < x_1 < x_2 < x_3 < x_6,$$

each of which conforms with the conditions $x_1 < x_2 < x_3$ and $x_4 < x_5 < x_6$.

Consider now the totally antisymmetric space eigenfunction $\psi_{\rm AS}$ of the Hamiltonian (1) with energy

$\varepsilon_0+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5$. It belongs to $\{6,0\}$, and is the ground state with such symmetry. In the interior of each subregion it is either all >0 or all <0. Now we compare $\psi_{\rm AS}$ with $\psi_{Y\infty}$ in any subregion of R_Y. They are both eigenfunctions of H. They both vanish on the boundary of the subregion. They both have no zeros in the interior of this subregion. Thus they must be proportional to each other, and have the same energy $\varepsilon_0+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5$. We can now continue $\psi_{Y\infty}$ into the full ∞^6 space by reflections on the four boundary planes (7). Thus in full ∞^6 space,

$$|\psi_{Y\infty}|=|\psi_{\rm AS}|. \tag{9}$$

This is a generalization of the early result of Girardeau[4].

$Y=\{4,2\}$

We now consider $Y=\{4,2\}$ and start with the product

$$[u_0(x_1)u_1(x_2)u_2(x_3)u_3(x_4)][u_0(x_5)u_1(x_6)],$$

arriving at the eigenfuction for $g=0$:

$$\begin{aligned}\psi_Y=&\det[u_0(x_1)u_1(x_2)u_2(x_3)u_3(x_4)]\\&\times\det[u_0(x_5)u_1(x_6)],\end{aligned} \tag{10}$$

with energy $(\varepsilon_0+\varepsilon_1+\varepsilon_2+\varepsilon_3)+(\varepsilon_0+\varepsilon_1)$.

This ψ_Y vanishes on the boundary of region R_Y,

$$R_Y:\ x_1<x_2<x_3<x_4,\ \text{ and }\ x_5<x_6, \tag{11}$$

bounded by four planes [Notice this R_Y is different from the R_Y for symmetry $\{3,3\}$]. Repeating the arguments for the case $Y=\{3,3\}$ we arrive at the conclusion that as $g\to\infty$, ψ_Y approaches a limit $\psi_{Y\infty}$, with $E\to\varepsilon_0+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5$. Reflections on the boundary planes extend $\psi_{Y\infty}$ into full ∞^6 space, and

$$|\psi_{Y\infty}|=|\psi_{\rm AS}|.$$

Thus we have

Theorem 3: The ground state of Hamiltonian (1) for symmetry Y is not degenerate. Denote its energy by E_Y. Then at $g=0$,

$$\begin{aligned}E_{\{3,3\}}&=(\varepsilon_0+\varepsilon_1+\varepsilon_2)+(\varepsilon_0+\varepsilon_1+\varepsilon_2),\quad (J=0),\\E_{\{4,2\}}&=(\varepsilon_0+\varepsilon_1+\varepsilon_2+\varepsilon_3)+(\varepsilon_0+\varepsilon_1),\quad (J=1),\\E_{\{5,1\}}&=(\varepsilon_0+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)+\varepsilon_0,\quad (J=2),\\E_{\{6,0\}}&=\varepsilon_0+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5,\quad (J=3).\end{aligned}$$

Furthermore, $E_{\{6,0\}}$ is independent of g. As g increases the other three all increase *monotonically* and approach $E_{\{6,0\}}$ as $g\to+\infty$.

To summarize, for the ground state wavefunction at $g=0$, we choose for each Y such functions as (4) and (9) which are products of two determinants. Then we follow these wavefunctions as $g\to+\infty$. They approach limits $\psi_{Y\infty}$ which satisfy equation (9), each vanishing on all 15 planes defined by the delta function interactions. Each $\psi_{Y\infty}$ is positive in one half of the $6!=720$ subregions, and negative in the other half. The $\psi_{Y\infty}$ for different Y's are orthogonal to each other. Notice they are *cusp-less on different sets of planes for different* Y's.

Furthermore, according to a theorem due to Lieb and Mattis,[5] for all finite g values,

$$E_{\{3,3\}}<E_{\{4,2\}}<E_{\{5,1\}}<E_{\{6,0\}}.$$

Obviously, these results can be generalized to any even or odd values of N.

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